ELSEVIER
Linear Algebra and its Applications 312 (2000) 35–71
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# On properties of Sylvester and Lyapunov operators[☆]

M. Konstantinov [a], V. Mehrmann [b,*], P. Petkov [c]

[a] *University of Architecture and Civil Engineering, 1 Hr. Smirnenski Blvd., 1421 Sofia, Bulgaria*
[b] *Fakultät für Mathematik, TU Chemnitz, D-09107 Chemnitz, Germany*
[c] *Department of Automatics, TU Sofia, 1756 Sofia, Bulgaria*

Received 26 July 1998; accepted 7 February 2000

Submitted by L. Rodman

**Abstract**

Sylvester and Lyapunov operators in real and complex matrix spaces are studied, which include as particular cases the operators arising in the theory of linear time-invariant systems. Let $\mathscr{M} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$ be a linear operator, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The operator $\mathscr{M}$ is *elementary* if there exist matrices $A \in \mathbb{F}^{p\times m}$ and $B \in \mathbb{F}^{q\times n}$, such that $\mathscr{M}[X] = AXB$. Each $\mathscr{M}$ can be represented as a sum of minimum number of elementary operators, called the *Sylvester index* of $\mathscr{M}$. An expression for the Sylvester index of a general linear operator $\mathscr{M}$ is given. An important tool here is a special permutation operator $\mathscr{V}_{p,m} : \mathbb{F}^{pq\times mn} \to \mathbb{F}^{pm\times nq}$ such that the image $\mathscr{V}_{p,m}(B^{\mathrm{T}} \otimes A)$ of the matrix of a non-zero elementary operator is equal to the rank 1 matrix $\mathrm{vec}[A]\mathrm{row}[B]$, where $\mathrm{vec}[X]$ and $\mathrm{row}[X]$ are the column-wise and row-wise vector representation of the matrix $X$. The application of $\mathscr{V}_{p,m}$ reduces a sum of Kronecker products of matrices to the standard product of two matrices. A linear operator $\mathscr{L} : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ is a *Lyapunov operator* if $(\mathscr{L}[X])^* = \mathscr{L}[X^*]$, where the star denotes transposition in the real case and complex conjugate transposition in the complex case. Characterisations and parametrisations of the sets of real and complex Lyapunov operators are given and their dimensions are found. Relevant *Lyapunov indexes* for Lyapunov operators are introduced and calculated. Similar results are given also for several classes of Lyapunov-like linear and pseudo-linear operators. The concept of *Lyapunov singular values* of a Lyapunov operator is introduced and the application of these values to the sensitivity and a posteriori error analysis of Lyapunov equations is discussed. © 2000 Elsevier Science Inc. All rights reserved.

☆ Work supported by grant 5/72 764 of VW Stiftung.
* Corresponding author.
*E-mail addresses:* mmk_fte@uacg.acad.bg (M. Konstantinov), mehrmann@mathematik.tu-chemnitz.de (V. Mehrmann), php@mbox.digsys.bg (P. Petkov).


PII: S 0 0 2 4 - 3 7 9 5 ( 0 0 ) 0 0 0 8 2 - 3

*AMS classification:* 15A24

*Keywords:* Linear matrix operators; Pseudo-linear matrix operators; Sylvester operators; Lyapunov operators; Singular values; Symmetric singular values; Sensitivity and error analysis

## 1. Introduction and notations

Linear matrix equations and linear matrix operators have been studied since the pioneering work of Sylvester and Kronecker [17,25,26], see also [1,20,21,29]. Now there are hundreds of papers, surveys and many books, e.g., [2,3,7,10,11,24,28] devoted to the analysis, existence, uniqueness and representation of the solution and also to the numerical algorithms and software to solve various types of linear matrix equations. Most of the existing results, however, are connected with particular classes of such matrix equations. In particular the problem of representing a general linear matrix operator as a sum of elementary operators seems to be not completely settled so far.

An important class of linear matrix equations are the Lyapunov equations. Since the fundamental work of Lyapunov on stability of motion, these matrix equations have been widely used in stability theory of differential equations [30], in the theory of linear-quadratic optimisation and filtering [19], in the perturbation analysis of linear and non-linear matrix equations [6,9,12,16] and other fields of pure and applied mathematics. This has motivated a continuous interest in both the theory and numerical treatment of Lyapunov operators and equations [4,5,8,22,23,27] and also recently in the context of the analysis and numerical simulation of descriptor systems via generalised Lyapunov equations [18]. Some general properties of finite-dimensional Lyapunov operators, however, have not been studied to a sufficient extent. In particular, the notion of the minimal singular value of a Lyapunov operator is sometimes misused. Introducing the new concept of Lyapunov singular values of a Lyapunov operator, some well-known estimates in the sensitivity theory of matrix equations may be substantially improved.

In this paper, we first investigate the general class of linear matrix operators, the Sylvester operators, and introduce the *Sylvester index* of such an operator as the minimum number of terms in which it can be represented as a sum of elementary (one-term) Sylvester operators. We then give an explicit expression for the index and derive a procedure for determining the representation of a general Sylvester operator as a sum of elementary Sylvester operators.

Furthermore, we study the general class of Lyapunov operators and give characterisations and parametrisations of the sets of real and complex Lyapunov operators. In particular the dimensions of these spaces are found. We also define and compute the *Lyapunov indexes*, which are relevant for Lyapunov operators. Similar results are derived for six more classes of Lyapunov-like operators. We then introduce the concept of *Lyapunov singular values* of Lyapunov operators and show their application to the perturbation and a posteriori error analysis of Lyapunov equations.

Several classes of Lyapunov-like linear and pseudo-linear operators are also considered.

We use the following notations.

- $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ – the sets of natural, real and complex numbers, $\jmath = \sqrt{-1}$;
- $\mathbb{F}^{m\times n}$ – the space of $m \times n$ matrices over $\mathbb{F}$, $\mathbb{F}^n = \mathbb{F}^{n\times 1}$, $\mathbb{F}_n = \mathbb{F}^{1\times n}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$;
- $A^{\mathrm{T}}$, $\overline{A}$ and $A^{\mathrm{H}} = \overline{A}^{\mathrm{T}}$ – the transpose, complex conjugate and complex conjugate transpose of a matrix $A$ ($A^*$ stands for $A^{\mathrm{T}}$ or $A^{\mathrm{H}}$);
- $\mathrm{Rg}[A]$ and $\mathrm{Ker}[A]$ – the image and kernel of the matrix $A$;
- $I_n$ – the unit $n \times n$ matrix;
- $0_{m\times n}$ or 0 – the zero $m \times n$ matrix or a zero matrix, whose size is clear from the context (if $m = 0$ or $n = 0$ the matrix $0_{m\times n}$ is void);
- $E_{i,j}(m, n) \in \mathbb{R}^{m\times n}$ – the matrix with a single non-zero entry, equal to 1, in position $(i, j)$;
- $\mathrm{tr}[A]$, $\det[A]$ and $\mathrm{rank}[A]$ – the trace, determinant and rank of the matrix $A$;
- $\|A\|_2 = \sigma_{\max}[A]$ – the spectral norm of the matrix $A$, where $\sigma_{\max}(A)$ is the maximum singular value of $A$;
- $\|A\|_{\mathrm{F}} = \sqrt{\mathrm{tr}(A^{\mathrm{H}}A)}$ – the Frobenius norm of the matrix $A$ (we use the same notation for the Frobenius norm of a linear operator);
- $\mathrm{diag}(a, \dots, b)$ – diagonal or block-diagonal matrix with elements or blocks $a, \dots, b$ on the main diagonal;
- $\mathrm{vec}_{m,n}[A] = \left[a_1^{\mathrm{T}}, \dots, a_n^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{F}^{mn}$ – the column-wise vector representation of the matrix $A = [a_1, \dots, a_n] \in \mathbb{F}^{m\times n}$, $a_j \in \mathbb{F}^m$. We also consider $\mathrm{vec}_{m,n}$ as a linear operator $\mathbb{F}^{m\times n} \to \mathbb{F}^{mn}$ (the dependence of $\mathrm{vec}_{m,n}$ on the matrix dimensions $m, n$ is usually omitted writing $\mathrm{vec}[A]$), $\mathrm{vec}_n = \mathrm{vec}_{n,n}$;
- $\mathrm{vec}_{m,n}^{-1} : \mathbb{F}^{mn} \to \mathbb{F}^{m\times n}$ – the inverse of $\mathrm{vec}_{m,n} : \mathbb{F}^{m\times n} \to \mathbb{F}^{mn}$ (here the dependence on at least one of the indexes $m, n$ cannot be omitted), $\mathrm{vec}_n^{-1} = \mathrm{vec}_{n,n}^{-1}$;
- $\Pi_{m,n} \in \mathbb{R}^{mn\times mn}$ – the vec-permutation matrix such that $\mathrm{vec}[M^{\mathrm{T}}] = \Pi_{m,n}\mathrm{vec}[M]$ for $M \in \mathbb{F}^{m\times n}$, $\Pi_n = \Pi_{n,n}$;
- $\mathrm{row}_{m,n}[A] = (\mathrm{vec}[A^{\mathrm{T}}])^{\mathrm{T}} = [\alpha_1, \dots, \alpha_m] = (\mathrm{vec}[A])^{\mathrm{T}}\Pi_{n,m} \in \mathbb{F}_{mn}$ – the row-wise vector representation of the matrix $A = [\alpha_1^{\mathrm{T}}, \dots, \alpha_m^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{F}^{m\times n}$, where $\alpha_i \in \mathbb{F}_n$;
- $\Omega(n, \mathbb{F}) \subset \mathbb{F}^{n^2\times n^2}$ – the set of all matrices $L \in \mathbb{F}^{n^2\times n^2}$ such that $L\Pi_n = \Pi_n\overline{L}$;
- $A \otimes B = [a_{i,j}B] \in \mathbb{F}^{pn\times mq}$ – the Kronecker (tensor) product of the matrices $A = [a_{i,j}] \in \mathbb{F}^{p\times m}$ and $B \in \mathbb{F}^{n\times q}$;
- $\sigma[A] = \{\sigma_1[A], \dots, \sigma_k[A]\} \subset \mathbb{R}_+$ – the set of singular values $\sigma_1[A] \geqslant \cdots \geqslant \sigma_k[A] \geqslant 0$ of the matrix $A \in \mathbb{F}^{m\times n}$, counted according to their algebraic multiplicities, where $k = \min\{m, n\}$;
- $\mathbf{GL}(n, \mathbb{F}) \subset \mathbb{F}^{n\times n}$ – the group of non-singular matrices;
- $\mathbf{Her}(n, \mathbb{F}) \subset \mathbb{F}^{n\times n}$ – the set of Hermitian matrices, satisfying $A^* = A$;

- $\nu_+[A]$, $\nu_-[A]$ and $\nu_0[A]$ – the number of positive, negative and zero eigenvalues of the matrix $A \in \mathbf{Her}(n, \mathbb{F})$;
- $\mathbf{Sher}(n, \mathbb{F}) \subset \mathbb{F}^{n\times n}$ – the set of skew-Hermitian matrices, satisfying $A^* = -A$;
- $\mathbf{Lin}(p, m, n, q, \mathbb{F})$ – the space of linear matrix (Sylvester) operators $\mathscr{M} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$. We abbreviate $\mathbf{Lin}(m, n, \mathbb{F}) = \mathbf{Lin}(m, m, n, n, \mathbb{F})$ and $\mathbf{Lin}(n, \mathbb{F}) = \mathbf{Lin}(n, n, n, n, \mathbb{F})$;
- $0_{p,m,n,q}$ and $1_{m,n}$ – the zero operator from $\mathbf{Lin}(p, m, n, q, \mathbb{F})$ and the unit operator from $\mathbf{Lin}(m, n, \mathbb{F})$, respectively;
- $\mathrm{Mat}(\mathscr{M}) = M \in \mathbb{F}^{pq\times mn}$ – the matrix representation of the operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$, defined by $\mathrm{vec}[\mathscr{M}[X]] = M\,\mathrm{vec}[X]$;
- $\mathbf{Lyap}(n, \mathbb{F}) \subset \mathbf{Lin}(n, \mathbb{F})$ – the set of Lyapunov operators $\mathscr{L}$, such that $(\mathscr{L}[X])^* = \mathscr{L}[X^*]$.

The singular values $\sigma_i(\mathscr{M})$ of an operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ are, by definition, the singular values $\sigma_i[M]$ of its matrix representation $M$, i.e., $\sigma_i(\mathscr{M}) = \sigma_i[M]$. For operators $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{F})$ we also define Lyapunov singular values $\widetilde{\sigma}_i(\mathscr{L})$.

As a rule we use square brackets to denote the values of functions of matrix arguments as in $X \mapsto \mathscr{M}[X]$, where $X$ is a matrix. The notation $m|n$ means that $m, n, n/m \in \mathbb{N}$. The abbreviation ":=" stands for "equal by definition"

## 2. Linear matrix operators

### 2.1. Basic concepts

Denote by $\mathbf{Lin}(p, m, n, q, \mathbb{F})$ the linear space of linear matrix operators $\mathscr{M} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$, i.e., $\mathscr{M}[X] \in \mathbb{F}^{p\times q}$, $X \in \mathbb{F}^{m\times n}$. In what follows a linear operator will often depend on a collection of $2r$ matrices

$$C := (A_1, B_1, \dots, A_r, B_r) \in \Sigma_r := (\mathbb{F}^{p\times m} \times \mathbb{F}^{n\times q})^r, \tag{1}$$

where $A_k \in \mathbb{F}^{p\times m}$, $B_k \in \mathbb{F}^{n\times q}$. To emphasize this dependence we write $\mathscr{E}_r(C) \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ for the operator itself and $\mathscr{E}_r(C)[X] \in \mathbb{F}^{p\times q}$ for its matrix value at a given $X$. Thus we have a family of operators $\{\mathscr{E}_r(C)\}_{C\in\Sigma_r}$ and $\mathscr{E}_r$ may be considered as a mapping

$$\mathscr{E}_r(\cdot)[\cdot] : \Sigma_r \times \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}, \tag{2}$$

quadratic in its first argument $C \in \Sigma_r$ and linear in its second argument $X \in \mathbb{F}^{m\times n}$.

**Definition 1.** The operator $\mathscr{E}_1(A, B) \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ such that

$$\mathscr{E}_1(A, B)[X] := AXB, \quad X \in \mathbb{F}^{m\times n},$$

where $A \in \mathbb{F}^{p\times m}$, $B \in \mathbb{F}^{n\times q}$, is called an *one-term*, or *elementary Sylvester operator* with a pair of generating matrices $(A, B)$.

The *zero operator* $0_{p,m,n,q} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ and the *identity operator* $1_{m,n} \in \mathbf{Lin}(m, n, \mathbb{F})$ are elementary Sylvester operators $\mathscr{E}_1(A, B)$ with pairs of generating matrices $(A, 0_{n\times q})$ (or $(0_{p\times m}, B)$) and $(I_m, I_n)$, respectively, where $A \in \mathbb{F}^{p\times m}$ (or $B \in \mathbb{F}^{n\times q}$) is arbitrary. A pair $(A, B)$, corresponding to the zero operator (with at least one of its components $A$ or $B$ being zero), is said to be a *trivial pair*.

A general linear matrix operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ may be determined as follows. Let $pq$ vectors

$$m_{i,j} \in \mathbb{F}^{mn}, \quad i = 1, \dots, p, \quad j = 1, \dots, q$$

be given. Define the linear functionals $\mu_{i,j} : \mathbb{F}^{m\times n} \to \mathbb{F}$ from

$$\mu_{i,j}[X] := m_{i,j}^{\mathrm{T}} \mathrm{vec}[X] \in \mathbb{F}, \quad X \in \mathbb{F}^{m\times n}.$$

Then the operator $\mathscr{M} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$, given by

$$\mathscr{M}[X] = \left[\mu_{i,j}[X]\right]_{i,j=1}^{p,q}$$

is a linear matrix operator. The matrix

$$M := \mathrm{Mat}(\mathscr{M}) \in \mathbb{F}^{pq\times mn}$$

associated with $\mathscr{M}$, is defined via the equality

$$\mathrm{vec}[\mathscr{L}[X]] = M\mathrm{vec}[X]$$

and hence

$$M = [m_{1,1}, m_{2,1}, \dots, m_{p,1}, m_{1,2}, m_{2,2}, \dots, m_{p,2}, \dots, m_{1,q}, m_{2,q}, \dots, m_{p,q}]^{\mathrm{T}}.$$

In this formulation a linear matrix operator $\mathscr{M}$ has no particular structure and may be identified with its matrix representation $M \in \mathbb{F}^{pq\times mn}$ according to the commutative diagram

$$\begin{array}{ccc} \mathbb{F}^{m\times n} & \xrightarrow{\mathscr{M}} & \mathbb{F}^{p\times q} \\ \downarrow \mathrm{vec} & & \downarrow \mathrm{vec} \\ \mathbb{F}^{mn} & \xrightarrow{M} & \mathbb{F}^{pq} \end{array}$$

(to recover $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ from a given $M \in \mathbb{F}^{pq\times mn}$ we also need one of the integers $p$ or $q$ and one of the integers $m$ or $n$). At the same time any particular value of a linear matrix operator may be expressed as a sum of matrix products. In this framework the specific structure of the operator may be revealed as an alternative to its representation as a general $pq \times mn$ matrix. This special structure is encoded in the mapping (2).

Let a matrix $2r$-tuple as defined in (1) be given. Consider a non-zero operator $\mathscr{E}_r(C) \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$, which is represented as a sum of $r$ non-zero elementary Sylvester operators $\mathscr{E}_1(A_k, B_k)$, i.e.,

$$\mathscr{E}_r(C)[X] := \sum_{k=1}^{r} \mathscr{E}_1(A_k, B_k)[X] = \sum_{k=1}^{r} A_k X B_k, \quad X \in \mathbb{F}^{m\times n}. \tag{3}$$

Operators of the form (3) are called *Sylvester operators*.

Each $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ may be represented in the form (3), i.e., $\mathscr{M} = \mathscr{E}_r(C)$ for some $r$ and $C$, although this may not be a trivial task.

Applying the vec operation to the expression for $\mathscr{E}_r(C)[X]$ we get

$$\mathrm{vec}[\mathscr{E}_r(C)[X]] = E_r(C)\mathrm{vec}[X], \tag{4}$$

where

$$E_r = E_r(C) := \mathrm{Mat}(\mathscr{E}_r(C)) = \sum_{k=1}^{r} B_k^{\mathrm{T}} \otimes A_k \in \mathbb{F}^{pq\times mn} \tag{5}$$

is the *matrix, associated with the operator* $\mathscr{E}_r(C)$. The matrix $E_r$ will also be referred to as the *matrix representation* (or briefly the *matrix*) of the operator $\mathscr{E}_r$. Every collection $C$ determines a unique Sylvester operator $\mathscr{E}_r(C)$ through (3) but the converse is of course not true.

Using the vec operator and its inverse, $\mathrm{vec}_{p,q}^{-1} : \mathbb{F}^{pq} \to \mathbb{F}^{p\times q}$, any operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ and its matrix representation $M \in \mathbb{F}^{pq\times mn}$ are related via the relations

$$\begin{aligned} &\mathrm{vec}[\mathscr{M}[X]] = M\mathrm{vec}[X], \\ &\quad \mathscr{M}[X] = \mathrm{vec}_{p,q}^{-1}[M\mathrm{vec}[X]], \quad X \in \mathbb{F}^{m\times n}. \end{aligned}$$

There exist different integers $r \in \mathbb{N}$ and infinitely many collections $C \in \Sigma_r$, such that $\mathscr{M}$ has a representation of type (3), i.e., $\mathscr{M} = \mathscr{E}_r(C)$ for some collection $C$, which satisfies the bilinear matrix equation

$$\sum_{k=1}^{r} B_k^{\mathrm{T}} \otimes A_k = M. \tag{6}$$

Obviously pairs $(A_k, B_k)$ and $(\mu_k A_k, B_k/\mu_k)$ give rise to the same Sylvester operator. Another possibility to get different representations of type (3) of the same operator is when the matrix $M_k := B_k^{\mathrm{T}} \otimes A_k$ of some elementary Sylvester operator $\mathscr{E}_1(A_k, B_k)$ is a linear combination of the matrices of other Sylvester operators.

**Example 1.** Given $A \in \mathbb{F}^{m\times m}$ and $B \in \mathbb{F}^{n\times n}$ such that

$$\begin{aligned} \mathscr{M}[X] &= AXB + AX + XB = (A + I_m)X(B + I_n) - X \\ &= AX(B + I_n) + XB = AX + (A + I_m)XB, \end{aligned}$$

we see that the operator $\mathscr{M} := \mathscr{E}_3(A, B, A, I_n, I_m, B) \in \mathbf{Lin}(m, n, \mathbb{F})$ may be represented in at least three more ways as $\mathscr{E}_2(A + I_m, B + I_n, I_m, -I_n)$, $\mathscr{E}_2(A, B + I_n, I_m, B)$ and $\mathscr{E}_2(A, I_n, A + I_m, B)$.

The above observations lead to the problem of representing an operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ as a sum of minimum number of elementary Sylvester operators.

**Definition 2.** The minimum number $\ell \in \mathbb{N}$, such that the non-zero operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ may be represented as a sum of $\ell$ elementary Sylvester operators, is said to be the *Sylvester index* of $\mathscr{M}$ and is denoted by $\mathrm{ind}_{p,m,n,q}(\mathscr{M})$. The zero operator is of Sylvester index 1. Any representation of $\mathscr{M}$ as a sum of minimum number of elementary operators is called a *condensed representation*. We also abbreviate $\mathrm{ind}_{m,n} := \mathrm{ind}_{m,m,n,n}$ and $\mathrm{ind}_n := \mathrm{ind}_{n,n,n,n}$.

It follows from this definition that a Sylvester operator is elementary if and only if it has Sylvester index 1.

Some comments on the Sylvester index are necessary. First of all in the notation for the index of an operator it is possible to omit the dependence on the dimensions, writing $\mathrm{ind}(\mathscr{M})$, since the dimensions $p, m, n, q$ are implicit in $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$. However, we prefer to indicate explicitly (with certain redundancy) the dependence of the Sylvester index of $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ on the dimensions $p, m, n, q$ in order to have a universal definition, applicable to both the operator $\mathscr{M}$ and its associated matrix $M$ as shown below. Indeed, the matrix representation $M$ of $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ is of size $pq \times mn$. Let $p', m', n', q'$ be any integers such that $p'q' = pq$ and $m'n' = mn$. Then $M$ is also the matrix, associated with another operator $\mathscr{M}' \in \mathbf{Lin}(p', m', n', q')$ with Sylvester index $\mathrm{ind}_{p',m',n',q'}(\mathscr{M}')$, which may be different from the Sylvester index $\mathrm{ind}_{p,m,n,q}(\mathscr{M})$ of $\mathscr{M}$. However, for a given matrix $M \in \mathbb{F}^{pq\times mn}$ and fixed integers $p, m, n, q$ (when $M$ is given, it is enough only $p$ and $m$ to be fixed), there is an unique linear operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ with $\mathrm{Mat}(\mathscr{M}) = M$. Hence it makes sense to define also the *Sylvester index* $\mathrm{ind}_{p,m,n,q}[M]$ for any matrix $M \in \mathbb{F}^{pq\times mn}$ as equal to the Sylvester index of the corresponding linear operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$.

Given $p_0, m_0 \in \mathbb{N}$ and a matrix $M \in \mathbb{F}^{p_0\times m_0}$, we may define the maximum (minimum) Sylvester index $\mathrm{ind}_{\max(\min)}[M]$ of $M$ as the maximum(minimum) value of $\mathrm{ind}_{p,m,n,q}[M]$ over all $p, m, n, q \in \mathbb{N}$, satisfying $pq = p_0, mn = m_0$.

We note also that $\mathrm{ind}_{1,1,\bullet,\bullet} = 1$. As may be expected, the Sylvester index of a matrix $M \in \mathbb{F}^{pq\times mn}$ is symmetric in the sense that

$$\mathrm{ind}_{p,m,n,q}[M] = \mathrm{ind}_{q,n,m,p}[M]$$

(see Proposition 4).

Definition 2 applies also to operators $\mathscr{E}_r(C)$ in the form (3) and here we write $\mathrm{ind}_{p,m,n,q}(\mathscr{E}_r(C))$. The Sylvester index of $\mathscr{E}_r(C)$ in (3) is at most $r$ but may be much less.

**Example 2.** The Sylvester index of the operator

$$\mathscr{M} := \mathscr{E}_4(A, B, A, F, E, B, E, F)$$

as in (3) is at most 4, but in fact it is equal to 1, since $\mathscr{M}$ is the elementary operator $\mathscr{E}_1(A + E, B + F)$.

In Examples 1 and 2 some of the elementary Sylvester operators were linear combinations of other elementary Sylvester operators in the representation (3). Such elementary operators may be removed from the representation of a general Sylvester operator according to the following proposition.

**Proposition 1.** *Let an operator $\mathscr{E}_r(C)$ as in* (3) *be given. Then*

$$\mathrm{ind}_{p,m,n,q}(\mathscr{E}_r(C)) \leqslant s := \mathrm{rank}[\mathrm{vec}[B_1^{\mathrm{T}} \otimes A_1], \ldots, \mathrm{vec}[B_r^{\mathrm{T}} \otimes A_r]]. \tag{7}$$

**Proof.** Suppose that $s < r$ (if $s = r$ there is nothing to prove, since $\mathrm{ind}_{p,m,n,q}(\mathscr{E}_r(C)) \leqslant r$). Let $c_j := \mathrm{vec}[B_j^{\mathrm{T}} \otimes A_j] \in \mathbb{F}^{pmnq}$ and assume w.l.o.g. that the vectors $c_1, \ldots, c_s$ are linearly independent. Then every $c_k$ with $k > s$ may be expressed as

$$c_k = \sum_{j=1}^{s} \lambda_{k,j} c_j, \quad \lambda_{k,j} \in \mathbb{F}.$$

Hence, for $k > s$ we have

$$(B_k^{\mathrm{T}} \otimes A_k)\mathrm{vec}[X] = \sum_{j=1}^{s} \lambda_{k,j} (B_j^{\mathrm{T}} \otimes A_j)\mathrm{vec}[X]$$

and

$$A_k X B_k = \sum_{j=1}^{s} \lambda_{k,j} A_j X B_j.$$

Substituting this expression in (3) we obtain

$$\mathscr{E}_r(C)[X] = \mathscr{E}_s(\alpha_1 A_1, B_1, \ldots, \alpha_s A_s, B_s)[X] = \sum_{k=1}^{s} \alpha_k A_k X B_k,$$

where $\alpha_k := \sum_{i=s+1}^{r} \lambda_{i,k}$. Hence $\mathrm{ind}_{p,m,n,q}(\mathscr{E}_r(C)) \leqslant s$ as claimed. $\square$

It follows from Definition 2 that we may assume that the representation of type (3) of a Sylvester operator is condensed, i.e., that $r = \mathrm{ind}_{p,m,n,q}(\mathscr{E}_r(C))$. Of course, starting from a given representation (3) it is not a trivial task to get a condensed one. Deleting linearly dependent terms in (3) as shown in the proof of Proposition 1 is in general not sufficient to obtain a condensed representation. We stress also that for Lyapunov operators a special symmetric representation is useful, see Section 3.

For $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ we have

$$\|\mathscr{M}[X]\|_{\mathrm{F}} = \|\mathrm{vec}[\mathscr{M}[X]]\|_2 \leqslant \|M\|_2 \|\mathrm{vec}[X]\|_2 = \|M\|_2 \|X\|_{\mathrm{F}}$$

with equality holding if $\mathrm{vec}[X]$ is a right singular vector of the matrix $M$, corresponding to its maximum singular value $\|M\|_2$. Hence, we may define a norm in $\mathbf{Lin}(p, m, n, q, \mathbb{F})$ as follows.

**Definition 3.** The (Frobenius) norm of $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ is

$$\|\mathscr{M}\|_{\mathrm{F}} := \max\{\|\mathscr{M}[X]\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1\} = \|M\|_2.$$

Other norms as

$$\|\mathscr{M}\|_{\alpha,\beta} := \max\{\|\mathscr{M}[X]\|_{\alpha} : \|X\|_{\beta} = 1\}, \quad \alpha, \beta \geqslant 1,$$

where $\|\cdot\|_{\alpha}$ is a Hölder norm, may also be used. Here convenient expressions for $\|\cdot\|_{\alpha,\beta}$ are known only for $\alpha = \beta = 2$ when $\mathscr{M}$ is the standard Lyapunov operator $X \mapsto A^*X + XA$ or $X \mapsto A^*XA - X$ of (generically) Sylvester index 2, see e.g. [6,9].

### *2.2. Representation of a linear matrix operator as a sum of elementary Sylvester operators*

Consider the problem of representing a general linear matrix operator $\mathscr{M}$ with associated matrix $M$ in the form (3). The dimension (real or complex) of $\mathbf{Lin}(p, m, n, q, \mathbb{F}) \simeq \mathbb{F}^{pq\times mn} \simeq \mathbb{F}^{pmnq}$ is $pmnq$. In particular, for each matrix $M \in \mathbb{F}^{pq\times mn}$ there exists $C \in \Sigma_r$ with $r = \mathrm{ind}_{p,m,n,q}(\mathscr{M})$, such that the associated matrix $E_r(C)$ of the operator $\mathscr{E}_r(C) \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ is equal to $M$, i.e., $E_r(C) = M$.

Given a matrix $M \in \mathbb{F}^{pq\times mn}$ and an integer $r \in \mathbb{N}$, the equation for determining $C$ of the form (6) is consistent if and only if $r \geqslant \mathrm{ind}_{p,m,n,q}[M]$. If it is consistent, then it is also underdetermined and has a multi-parameter family of solutions.

Relation (6) may be considered also as an equation for both $r \in \mathbb{N}$ and $C \in \Sigma_r$. A particular solution is obtained as follows. Partition the matrix $M \in \mathbb{F}^{pq\times mn}$ into $nq$ blocks of size $p \times m$ as

$$M = [M_{i,j}], \quad M_{i,j} \in \mathbb{F}^{p\times m}, \quad i = 1, \ldots, q, \quad j = 1, \ldots, n. \tag{8}$$

Then $M$ may be written as

$$M = \sum_{i,j=1}^{q,n} E_{i,j}(q, n) \otimes M_{i,j}.$$

Therefore, in view of (6), a possible solution for $C$ is

$$A_k = M_{i,j}, \quad B_k = E_{j,i}(n, q), \quad k = k(i, j) := i + (j - 1)q,$$

in which the number of non-trivial pairs $(A_k, B_k)$ is the number of non-zero blocks $M_{i,j}$ of $M$, which is at most $nq$. Thus the resulting operator $\mathscr{E}_r(C)$ and hence $\mathscr{M}$ are of Sylvester index at most $nq$. A similar argument for the transposed operator from $\mathbf{Lin}(q, n, m, p, \mathbb{F})$ shows that this index is at most $pm$. Thus we have proved the following result.

**Proposition 2.** *The Sylvester index of the operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ satisfies*

$$\mathrm{ind}_{p,m,n,q}(\mathscr{M}) \leqslant \min\{pm, nq\}.$$

A much stronger assertion is given in Proposition 4.

Next we calculate the Sylvester index of a linear operator and construct a representation of type (3). For this purpose we introduce a special linear matrix operator $\mathscr{V}_{p,m}$, defined on matrix spaces $\mathbb{F}^{p_0\times m_0}$ when $p|p_0$ and $m|m_0$.

Let $p, m, p_0, m_0 \in \mathbb{N}$ be given integers such that $p|p_0$ and $m|m_0$. Then each matrix $Z \in \mathbb{F}^{p_0\times m_0}$ may be partitioned into $nq$ blocks $Z_{i,j}$ of size $p \times m$, where $q := p_0/p$, $n := m_0/m$:

$$Z = \begin{bmatrix} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,n} \\ Z_{2,1} & Z_{2,2} & \cdots & Z_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{q,1} & Z_{q,2} & \cdots & Z_{q,n} \end{bmatrix}, \quad Z_{i,j} \in \mathbb{F}^{p\times m}.$$

Set $z_{i,j} := \mathrm{vec}[Z_{i,j}]$ and define the linear operator

$$\mathscr{V}_{p,m} : \mathbb{F}^{pq\times mn} \to \mathbb{F}^{pm\times nq} \tag{9}$$

from

$$\begin{aligned} &\mathscr{V}_{p,m}[Z] \\ &\quad := [z_{1,1}, z_{2,1}, \ldots, z_{q,1}, z_{1,2}, z_{2,2}, \ldots, z_{q,2}, \ldots, z_{1,n}, z_{2,n}, \ldots, z_{q,n}]. \end{aligned} \tag{10}$$

We see that $\mathscr{V}_{p,m}$ is a permutation operator, satisfying

$$\begin{aligned} \mathscr{V}_{p,q} \circ \mathscr{V}_{p,m} &= \mathscr{V}_{p,q} \circ \mathscr{V}_{p,n} = \mathscr{V}_{q,p} \circ \mathscr{V}_{q,m} \\ &= \mathscr{V}_{q,p} \circ \mathscr{V}_{q,n} = 1_{p,m,n,q}. \end{aligned} \tag{11}$$

Some other properties of the operator $\mathscr{V}_{p,m}$ are described in Proposition 3.

**Proposition 3.** *Let $M \in \mathbb{F}^{pq\times mn}$, $A \in \mathbb{F}^{p\times m}$ and $B = [b_{i,j}] \in \mathbb{F}^{n\times q}$. Then*

$$\begin{aligned} \mathscr{V}_{1,1}[M] &= (\mathrm{vec}[M])^{\mathrm{T}}, \\ \mathscr{V}_{1,mn}[M] &= M^{\mathrm{T}}, \\ \mathscr{V}_{p,m}[M] &= (\mathscr{V}_{q,n}[\Pi_{n,q} M \Pi_{n,m}])^{\mathrm{T}}, \\ \mathscr{V}_{pq,1}[M] &= M, \\ \mathscr{V}_{pq,mn}[M] &= \mathrm{vec}[M], \\ \mathscr{V}_{p,m}[\Pi_{m,p}] &= \Pi_{m,p} \end{aligned} \tag{12}$$

*and*

$$\mathscr{V}_{p,m}[B^{\mathrm{T}} \otimes A] = \mathrm{row}[B] \otimes \mathrm{vec}[A] = \mathrm{vec}[A]\mathrm{row}[B]. \tag{13}$$

**Proof.** Relations (12) follow from the definition of $\mathscr{V}_{p,m}$. To prove (13) we note that

$$B^{\mathrm{T}} \otimes A = \sum_{i,j=1}^{n,q} b_{i,j} E_{i,j}(q,n) \otimes A$$

$$= \begin{bmatrix} b_{1,1}A & b_{2,1}A & \dots & b_{n,1}A \\ b_{1,2}A & b_{2,2}A & \dots & b_{n,2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1,q}A & b_{2,q}A & \dots & b_{n,q}A \end{bmatrix}$$

and hence

$$\begin{aligned} \mathscr{V}_{p,m}(B^{\mathrm{T}} \otimes A) = {} & [b_{1,1}\mathrm{vec}[A], b_{1,2}\mathrm{vec}[A], \dots, b_{1,q}\mathrm{vec}[A], \\ & b_{2,1}\mathrm{vec}[A], b_{2,2}\mathrm{vec}[A], \dots, b_{2,q}\mathrm{vec}[A], \dots, \\ & b_{n,1}\mathrm{vec}[A], b_{n,2}\mathrm{vec}[A], \dots, b_{n,q}\mathrm{vec}[A]] \\ = {} & \mathrm{row}[B] \otimes \mathrm{vec}[A] = \mathrm{vec}[A]\mathrm{row}[B] \end{aligned}$$

as claimed. □

A similar operator $\mathscr{V}^*_{p,m} : \mathbb{F}^{pq\times mn} \to \mathbb{F}^{nq\times pm}$ may also be defined such that the rows of $\mathscr{V}^*_{p,m}[Z]$ are the row-wise vector representations $\mathrm{row}[Z_{i,j}]$ of the $p \times m$ blocks $Z_{i,j}$ of $Z$, taken in row-wise the order

$$(1,1)(1,2)\cdots(1,n)(2,1)(2,2)\cdots(2,n)\cdots(q,1)(q,2)\cdots(q,n).$$

For definiteness we shall use the operator $\mathscr{V}_{p,m}$. It allows the reduction of a sum of Kronecker products of matrices into a product of two matrices. Thus in particular one may solve efficiently Eq. (6).

Suppose that $M \in \mathbb{F}^{pq\times mn}$ is the matrix representation of the operator $\mathscr{M} \in \mathbf{Lin}(p,m,n,q,\mathbb{F})$, partitioned as in (8), and set

$$M^{\#} := \Pi_{p,q} M \Pi_{n,m} = [M^{\#}_{k,l}],\ M^{\#}_{k,l} \in \mathbb{F}^{q\times n}, \quad k = 1,\dots,p,\ \ l = 1,\dots,m.$$

Using the operator $\mathscr{V}_{\bullet,\bullet}$ define the matrices

$$\begin{aligned} \mathbf{M} &:= \mathscr{V}_{p,m}[M] \\ &= \left[\mathrm{vec}[M_{1,1}],\dots,\mathrm{vec}[M_{q,1}],\dots,\mathrm{vec}[M_{1,n}],\dots,\mathrm{vec}[M_{q,n}]\right] \in \mathbb{F}^{pm\times qn} \end{aligned}$$

and

$$\begin{aligned} \mathbf{M}^{\#} &:= \mathscr{V}_{q,n}[M^{\#}] \\ &= \left[\mathrm{vec}[M^{\#}_{1,1}],\dots,\mathrm{vec}[M^{\#}_{p,1}],\dots,\mathrm{vec}[M^{\#}_{1,m}],\dots,\mathrm{vec}[M^{\#}_{p,m}]\right] \in \mathbb{F}^{qn\times pm}. \end{aligned}$$

Now we can determine the Sylvester index of an arbitrary operator $\mathscr{M} \in \mathbf{Lin}(p,m,n,q,\mathbb{F})$ and construct a matrix collection $C \in \Sigma_r$ such that $\mathscr{M} = \mathscr{E}_r(C)$.

**Proposition 4.** *Let $M \in \mathbb{F}^{pq\times mn}$ be the matrix representation of the operator $\mathscr{M} \in \mathbf{Lin}(p,m,n,q,\mathbb{F})$. Then*

$$\mathrm{ind}_{p,m,n,q}(\mathscr{M}) = \mathrm{ind}_{p,m,n,q}[M] = \mathrm{ind}_{q,n,m,p}[M] = \max\{1, \rho[\mathbf{M}]\},$$

*where*

$$\rho[M] := \mathrm{rank}[\mathbf{M}] = \mathrm{rank}[\mathbf{M}^{\#}].$$

**Proof.** It follows from Proposition 3 that for given $r \in \mathbb{N}$ the Eq. (6) for $C = (A_1, B_1, \ldots, A_r, B_r)$ may be written as a bilinear equation

$$\mathbf{AB} = \mathbf{M} \tag{14}$$

in the unknown matrices

$$\mathbf{A} := [\mathrm{vec}[A_1], \mathrm{vec}[A_2], \ldots, \mathrm{vec}[A_r]] \in \mathbb{F}^{pm\times r},$$
$$\mathbf{B} := [\mathrm{vec}[B_1], \mathrm{vec}[B_2], \ldots, \mathrm{vec}[B_r]]^{\mathrm{T}} \Pi_{q,n} = \begin{bmatrix} \mathrm{row}[B_1] \\ \mathrm{row}[B_2] \\ \vdots \\ \mathrm{row}[B_r] \end{bmatrix} \in \mathbb{F}^{r\times nq}. \tag{15}$$

Eqs. (14) and (15) are fundamental for determining the indexes as well as for the construction of the linear matrix operator $\mathscr{M}$ as a sum of elementary operators, provided the matrix $M$ of $\mathscr{M}$ is given.

Let $\Theta_r[M] \subset \mathbb{F}^{pm\times r} \times \mathbb{F}^{r\times nq}$ be the set of solutions of (14). We shall show that $\Theta_r[M] \neq \emptyset$ if and only if $r \geqslant \rho[M]$ and hence Eq. (14) is solvable for $r = \rho[M]$. The proof is constructive and provides explicit expressions for $\Theta_{\rho[M]}[M]$.

In the trivial case $\mathscr{M} = 0_{p,m,n,q}$ we have $r = 1$ by definition and the solution of (14) may be taken as $(\mathbf{A}, 0_{1\times nq})$ or $(0_{pm\times 1}, \mathbf{B})$ with $\max\{pm, nq\}$ free parameters. Hence

$$\Theta_1[0_{p,m,n,q}] = \left(\mathbb{F}^{pm} \times \{0_{1\times nq}\}\right) \cup \left(\{0_{pm\times 1}\} \times \mathbb{F}_{nq}\right).$$

Consider the general case $\mathscr{M} \neq 0_{p,m,n,q}$. It follows from (14) that

$$\rho[M] \leqslant \min\{\mathrm{rank}[\mathbf{A}], \mathrm{rank}[\mathbf{B}]\} \leqslant r.$$

We shall prove that if $r = \rho[M]$, then (14) is explicitly solved. Consider the three possible cases.

1. If $r = \rho[M] = pm \leqslant nq$ then the solution set is

$$\Theta_r[M] = \left\{(P, P^{-1}\mathbf{M}) : P \in \mathbf{GL}(pm, \mathbb{F})\right\}.$$

2. If $r = \rho[M] = nq < mn$ the solution set is

$$\Theta_r[M] = \left\{(\mathbf{M}P^{-1}, P) : P \in \mathbf{GL}(nq, \mathbb{F})\right\}.$$

3. If $r = \rho[M] < \min\{pm, nq\}$, then $\mathbf{M}$ may be decomposed as

$$\mathbf{M} = U \mathrm{diag}\left(I_r, 0_{(pm-r)\times(nq-r)}\right) V^{-1},$$

where $U \in \mathbf{GL}(pm, \mathbb{F})$, $V \in \mathbf{GL}(nq, \mathbb{F})$.
Thus the solution set may be represented as

$$\Theta_r[M] = \left\{\left(U^{-1}\begin{bmatrix} P \\ 0_{(pm-r)\times r}\end{bmatrix}, \left[P^{-1}, 0_{r\times(nq-r)}\right] V\right) : P \in \mathbf{GL}(r, \mathbb{F})\right\}.$$

Similar arguments hold true for the transposed operator with a matrix $M^{\#}$, showing that $\mathrm{ind}_{p,m,n,q}[M] = \mathrm{ind}_{q,n,m,p}[M]$. Note finally that $\mathbf{M}^{\#} = \mathbf{M}^{\mathrm{T}}$, see Proposition 3. □

We see from the proof of Proposition 4 that in the non-trivial case $\mathscr{M} \neq 0_{p,m,n,q}$ the set of all collections $C$ in the condensed representation of $\mathscr{M}$ is isomorphic to $\mathbf{GL}(r, \mathbb{F})$, where $r$ is the Sylvester index of $\mathscr{M}$. Hence it is an open algebraic variety (of real or complex dimension $r^2$) in the corresponding Zariski topology.

**Example 3.** Consider the *transposition operator* $\mathscr{T}_{m,n} \in \mathbf{Lin}(n, m, n, m, \mathbb{F})$, acting as $\mathscr{T}_{m,n}[X] = X^{\mathrm{T}}$. The matrix representation of $\mathscr{T}_{m,n}$ is $\Pi_{m,n}$. Since $\mathscr{V}_{n,m}[\Pi_{m,n}] = \Pi_{m,n}$ (see Proposition 3) and $\mathrm{rank}[\Pi_{m,n}] = mn$, we see that $\mathrm{ind}_{n,m,n,m}(\mathscr{T}_{m,n}) = mn$. In particular [11] we have

$$X^{\mathrm{T}} = \sum_{i,j=1}^{n,m} E_{i,j}(n, m) X E_{i,j}(n, m).$$

Consider the case when $mn = pq$ and the operator $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$ is invertible, i.e., its associated matrix $M \in \mathbb{F}^{mn \times mn}$ is non-singular. For some classes of invertible operators it may be shown that

$$\mathrm{ind}_{p,m,n,q}[M] = \mathrm{ind}_{m,p,q,n}[M^{-1}]. \tag{16}$$

It is interesting to determine whether (16) holds for all invertible operators $\mathscr{M} \in \mathbf{Lin}(p, m, n, q, \mathbb{F})$.

## 3. Lyapunov operators

### 3.1. Real Lyapunov operators

An important class of linear operators are the Lyapunov operators, which are automorphisms in $\mathbb{F}^{n \times n}$. In this section, we consider the class of real Lyapunov operators in $\mathbf{Lin}(n, \mathbb{R})$.

**Definition 4.** An operator $\mathscr{L} \in \mathbf{Lin}(n, \mathbb{R})$ is called a *real Lyapunov operator* if

$$(\mathscr{L}[X])^{\mathrm{T}} = \mathscr{L}[X^{\mathrm{T}}], \quad X \in \mathbb{R}^{n \times n}.$$

We denote by $\mathbf{Lyap}(n, \mathbb{R}) \subset \mathbf{Lin}(n, \mathbb{R})$ the set of real Lyapunov operators.

It follows from Definition 4 that

$$X = X^{\mathrm{T}} \Rightarrow \mathscr{L}[X] = (\mathscr{L}[X])^{\mathrm{T}},$$
$$X = -X^{\mathrm{T}} \Rightarrow \mathscr{L}[X] = -(\mathscr{L}[X])^{\mathrm{T}}$$

provided $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$. Hence the subspaces $\mathbf{Her}(n, \mathbb{R})$ of symmetric and $\mathbf{Sher}(n, \mathbb{R})$ of skew-symmetric real matrices are invariant subspaces for Lyapunov operators from $\mathbf{Lyap}(n, \mathbb{R})$ (see also [4], where the particular case $\mathscr{L}[X] = A^*X + XA$ has been considered).

Below we shall need the operator $\mathscr{V}_n := \mathscr{V}_{n,n} : \mathbb{F}^{n^2\times n^2} \to \mathbb{F}^{n^2\times n^2}$, defined by (9) and (10) for $p = m = n = q$, which in the given case is an involutary permutation, $\mathscr{V}_n^2 = 1_{n,n}$.

Obviously $\mathbf{Lyap}(n, \mathbb{R})$ itself is a linear subspace of $\mathbf{Lin}(n, \mathbb{R})$, which may be characterised by Proposition 5.

**Proposition 5.** *The following four statements are equivalent*:

(i) $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$.

(ii) *There exists* $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{R})$, *such that*

$$\mathscr{L}[X] = \mathscr{M}[X] + (\mathscr{M}[X^{\mathrm{T}}])^{\mathrm{T}}, \quad X \in \mathbb{F}^{n\times n},$$

*i.e.*,

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left(A_k X B_k + B_k^{\mathrm{T}} X A_k^{\mathrm{T}}\right)$$

*or equivalently*

$$L := \mathrm{Mat}(\mathscr{L}) = \sum_{k=1}^{r} \left(B_k^{\mathrm{T}} \otimes A_k + A_k \otimes B_k^{\mathrm{T}}\right),$$

*where* $A_k, B_k \in \mathbb{R}^{n\times n}$ *are given matrices.*

(iii) $L \in \Omega(n, \mathbb{R})$, *where* $\Omega(n, \mathbb{R})$ *is the subspace of real* $n^2 \times n^2$ *matrices* $L$, *satisfying the equation* $\Pi_n L = L\Pi_n$.

(iv) *The matrix* $\mathbf{L} := \mathscr{V}_n(L)$ *is symmetric.*

**Proof.** The equivalence between (i) and (ii) follows from the definitions. To prove (iii) we perform vec operation on both sides of the characteristic equation $(\mathscr{L}[X])^{\mathrm{T}} = \mathscr{L}[X^{\mathrm{T}}]$ of the Lyapunov operator $\mathscr{L}$ with associated matrix $L$, which gives

$$\begin{aligned}
\mathrm{vec}\left[(\mathscr{L}[X])^{\mathrm{T}}\right] &= \mathrm{vec}\left[\mathscr{L}[X^{\mathrm{T}}]\right],\\
\Pi_n \mathrm{vec}[\mathscr{L}[X]] &= L\mathrm{vec}[X^{\mathrm{T}}],\\
\Pi_n L\mathrm{vec}[X] &= L\Pi_n \mathrm{vec}[X]
\end{aligned}$$

for all $X \in \mathbb{R}^{n\times n}$ and hence $\Pi_n L = L\Pi_n$.

To prove (iv) note that the relation

$$\sum_{k=1}^{r} \left(B_k^{\mathrm{T}} \otimes A_k + A_k \otimes B_k^{\mathrm{T}}\right) = L$$

from (ii) is an equation for the matrices $A_1, B_1, \ldots, A_r, B_r$, similar to (6). After some calculations we get the following counterpart of the bilinear equation (14) and (15):

$$\mathbf{AB} + (\mathbf{AB})^{\mathrm{T}} = \mathbf{L} \tag{17}$$

and hence the matrix $\mathbf{L}$ is symmetric. □

Representations of $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$ as in Proposition 5(ii) usually arise in the theory of continuous-time standard and descriptor dynamical systems. They involve $2r$ terms and may not be condensed in the sense of Definition 2. In particular, the representation of the Lyapunov operator $X \mapsto DXD^{\mathrm{T}}$ (of Sylvester index 1) in the form (ii) requires two terms, e.g. $r = 1$ and $A_1 = D$, $B_1 = D^{\mathrm{T}}/2$.

As in the case of a general Sylvester operator $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{F})$, the real Lyapunov operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$ may be represented in a condensed form as a sum of $\mathrm{ind}_n(\mathscr{L})$ elementary linear operators (not necessarily Lyapunov) but in this case the formal symmetry in Proposition 5(ii) may be lost. To preserve this symmetry, characterising Lyapunov operators, we shall introduce a special symmetric representation.

Each non-zero operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$ admits the following two symmetric representations.

The *continuous-time representation* is of the form

$$\mathscr{L}[X] = \sum_{k=1}^{\ell_{\mathrm{c}}} \left( A_k X B_k + B_k^{\mathrm{T}} X A_k^{\mathrm{T}} \right), \quad X \in \mathbb{R}^{n \times n}, \tag{18}$$

while the *discrete-time representation* is

$$\mathscr{L}[X] = \sum_{j=1}^{\ell_{\mathrm{d}}} \varepsilon_j D_j X D_j^{\mathrm{T}}, \quad X \in \mathbb{R}^{n \times n}, \tag{19}$$

where $\varepsilon_j = \pm 1$ and $D_j, A_k, B_k \in \mathbb{R}^{n \times n}$. Obviously $2\ell_{\mathrm{c}} \geqslant \mathrm{ind}_n(\mathscr{L})$ and $\ell_{\mathrm{d}} \geqslant \mathrm{ind}_n(\mathscr{L})$.

Mixed representations as

$$\mathscr{L}[X] = DXD^{\mathrm{T}} + A^{\mathrm{T}}X + XA$$

may be reduced to some of the above two types (18) or (19).

We see that when symmetry is desired, the Sylvester index and the condensed form according to Definition 2 may not be relevant to Lyapunov operators. This motivates the introduction of the following modified concepts.

**Definition 5.** The representations (18) and (19) of $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$ are said to be *cl-condensed* and *dl-condensed* respectively, if there are no representations of $\mathscr{L}$ of the corresponding types with less terms. The numbers $\mathrm{clind}_n(\mathscr{L}) := 2\ell_{\mathrm{c}}$ and $\mathrm{dlind}_n(\mathscr{L}) := \ell_{\mathrm{d}}$ in the cl-condensed representation and in the dl-condensed representation are called the *continuous-time Lyapunov index* and the *discrete-time Lyapunov index* of $\mathscr{L}$.

**Example 4.** Let $\lambda_1 > 0$ and $\lambda_2$ be reals. Then the operator $\mathscr{L} \in \mathbf{Lin}(2, \mathbb{R})$, defined from

$$\mathscr{L}[X] := \begin{bmatrix} 2\lambda_1 x_{1,1} & (\lambda_1 + \lambda_2)x_{1,2} \\ (\lambda_1 + \lambda_2)x_{2,1} & 2\lambda_2 x_{2,2} \end{bmatrix}, \quad X = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$$

has both its Lyapunov indexes equal to 2. It admits the following cl-condensed $\mathscr{L}[X] = AX + XA$ and dl-condensed $\mathscr{L}[X] = D_1 X D_1 - D_2 X D_2$ representations, where $A := \operatorname{diag}(\lambda_1, \lambda_2)$ and

$$D_1 := \operatorname{diag}\left(\sqrt{2\lambda_1}, \frac{\lambda_1 + \lambda_2}{\sqrt{2\lambda_1}}\right), \quad D_2 := \operatorname{diag}\left(0, \frac{\lambda_1 - \lambda_2}{\sqrt{2\lambda_1}}\right).$$

Explicit expressions for the Lyapunov indexes of Lyapunov operators are given below. Obviously

$$\operatorname{ind}_n(\mathscr{L}) \leqslant \min\{\operatorname{clind}_n(\mathscr{L}), \operatorname{dlind}_n(\mathscr{L})\}.$$

In fact we shall show that the Sylvester index of a Lyapunov operator $\mathscr{L}$ is equal to its discrete-time Lyapunov index, i.e. that $\operatorname{ind}_n(\mathscr{L}) = \operatorname{dlind}_n(\mathscr{L})$.

**Proposition 6.** *The continuous-time and the discrete-time Lyapunov indexes of the non-zero operator* $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$ *are determined from*

$$\operatorname{clind}_n(\mathscr{L}) = 2\max\{\nu_+(\mathbf{L}), \nu_-(\mathbf{L})\},$$
$$\operatorname{dlind}_n(\mathscr{L}) = \nu_+(\mathbf{L}) + \nu_-(\mathbf{L}) = \operatorname{rank}[\mathbf{L}].$$

It follows from Proposition 6 that the Sylvester and the continuous-time Lyapunov index of an operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$ coincide, i.e.

$$\operatorname{ind}_n(\mathscr{L}) = \operatorname{dlind}_n(\mathscr{L}) \leqslant \operatorname{clind}_n(\mathscr{L}).$$

**Proof.** Consider first the continuous-time case and set $\mathbf{C} = \mathbf{AB}$ in Eq. (17). Hence the number $r := \operatorname{clind}_n(\mathscr{L})/2$ may be computed from

$$r = \min\left\{\operatorname{rank}[\mathbf{C}] : \mathbf{C} \in \mathbb{R}^{n^2 \times n^2}, \mathbf{C} + \mathbf{C}^{\mathrm{T}} = \mathbf{L}\right\}.$$

Denote $\alpha := \nu_+[\mathbf{L}]$, $\beta := \nu_-[\mathbf{L}]$ and $\gamma := \alpha + \beta$. Supposing without loss of generality that $\alpha \geqslant \beta$ we shall show that $r = \alpha$. Indeed, there exists $P \in \mathbf{GL}(n^2, \mathbb{R})$ such that the matrix $\mathbf{L}$ is factorised as $\mathbf{L} = P\varDelta_{\mathbf{L}}P^{\mathrm{T}}$, $\varDelta_{\mathbf{L}} := \operatorname{diag}(2I_\alpha, -2I_\beta, 0_{n^2-\gamma})$. Setting $\mathbf{C} = PYP^{\mathrm{T}}$ we obtain that $r$ is the minimum of the ranks of the matrices $Y$, such that $Y + Y^{\mathrm{T}} = \varDelta_{\mathbf{L}}$. The general form of $Y$ is

$$Y = \begin{bmatrix} I_\alpha + Y_{1,1} & -Y_{2,1}^{\mathrm{T}} & -Y_{3,1}^{\mathrm{T}} \\ Y_{2,1} & -I_\beta + Y_{2,2} & -Y_{3,2}^{\mathrm{T}} \\ Y_{3,1} & Y_{3,2} & Y_{3,3} \end{bmatrix}, \tag{20}$$

where the matrices $Y_{1,1} \in \mathbf{Sher}(\alpha, \mathbb{R})$, $Y_{2,1} \in \mathbb{R}^{\beta \times \alpha}$, $Y_{3,1} \in \mathbb{R}^{(n^2-\gamma)\times\alpha}$, $Y_{2,2} \in \mathbf{Sher}(\beta, \mathbb{R})$, $Y_{3,2} \in \mathbb{R}^{(n^2-\gamma)\times\beta}$, $Y_{3,3} \in \mathbf{Sher}(n^2 - \gamma, \mathbb{R})$ are arbitrary. Suppose w.l.o.g. that

$\alpha \geqslant \beta$. The eigenvalues $\lambda(I_\alpha + Y_{1,1})$ of the matrix $I_\alpha + Y_{1,1}$ are equal to $1 + \lambda(Y_{1,1})$. In turn, $Y_{1,1}$ has its eigenvalues on the imaginary axis, i.e., the eigenvalues of $I_\alpha + Y_{1,1}$ have real part 1. Hence the diagonal block $I_\alpha + Y_{1,1}$ of $Y$ in (20) is non-singular and $\operatorname{rank}[Y] \geqslant \operatorname{rank}[I_\alpha + Y_{1,1}] = \alpha$. Moreover, for certain $Y$ the equality $\operatorname{rank}[Y] = \alpha$ is achieved. To see this take $Y_{1,1}$, $Y_{3,1}$, $Y_{2,2}$ and $Y_{3,3}$ as zero matrices, and let $Y_{2,1} := [I_\beta, 0_{\beta\times(\alpha-\beta)}]$. Then $Y_{2,1}Y_{2,1}^{\mathrm{T}} = I_\beta$ and hence

$$\begin{bmatrix} I_\alpha & 0 & 0 \\ -Y_{2,1} & I_\beta & 0 \\ 0 & 0 & I_{n^2-\gamma} \end{bmatrix} Y = \begin{bmatrix} I_\alpha & -Y_{2,1}^{\mathrm{T}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which yields $\operatorname{rank}[Y] = \alpha$. Therefore we may find matrices $\mathbf{A}$, $\mathbf{B}$, satisfying $\mathbf{AB} + (\mathbf{AB})^{\mathrm{T}} = \mathbf{L}$ with $r = \alpha$. Since the continuous-time representation of a Lyapunov operator has $2r$ terms, we have proved the first part of the proposition.

Consider now the discrete-time case. Denote

$$\mathbf{D} := [\operatorname{vec}[D_1], \ldots, \operatorname{vec}[D_r]] \in \mathbb{R}^{n^2\times r}$$

and let $E \in \mathbf{GL}(r, \mathbb{R})$ be a diagonal matrix with elements $\varepsilon_j = \pm 1$ on the diagonal. Then the equation

$$\sum_{k=1}^{r} \varepsilon_k (D_k \otimes D_k) = L$$

for the matrices $D_1, \ldots, D_k$ becomes $\mathbf{D}E\mathbf{D}^{\mathrm{T}} = \mathbf{L}$. We have $r \geqslant \gamma = \operatorname{rank}[\mathbf{L}]$. Consider again the factorisation $\mathbf{L} = P\varDelta_{\mathbf{L}}P^{\mathrm{T}}$. Partitioning the matrix $P$ as $P = [P_1, P_2]$, $P_1 \in \mathbb{R}^{n^2\times\gamma}$, we may choose $r = \gamma$ and $\mathbf{D} = P_1$, $E = \operatorname{diag}(I_\alpha, -I_\beta)$. □

According to parts (i) and (iii) of Proposition 5, a matrix $L \in \mathbb{R}^{n^2\times n^2}$ is the matrix representation of a Lyapunov operator if and only if it has the symmetry property $\varPi_n L = L\varPi_n$, or, equivalently, $L = \varPi_n L \varPi_n$. This leads to the following proposition.

**Proposition 7.** *The subspace $\varOmega(n, \mathbb{R}) \subset \mathbb{R}^{n^2\times n^2}$ of matrix representations of real Lyapunov operators is isomorphic to the subspace*

$$\operatorname{Ker}\left[I_{n^2} \otimes \varPi_n - \varPi_n \otimes I_{n^2}\right] = \operatorname{Ker}\left[\varPi_n \otimes \varPi_n - I_{n^4}\right] \subset \mathbb{R}^{n^4}. \tag{21}$$

**Proof.** Multiplying the last equation on the left with $\varPi_n$ and taking into consideration that $\varPi_n^2 = I_{n^2}$, we also get $L = \varPi_n L \varPi_n$. The characterisation of $\varOmega(n, \mathbb{R})$ by the subspace (21) is obtained taking the vec operation on both sides of the equalities $\varPi_n L - L\varPi_n = 0_{n^2\times n^2}$ and $\varPi_n L \varPi_n - L = 0_{n^2\times n^2}$, namely $(I_{n^2} \otimes \varPi_n - \varPi_n \otimes I_{n^2})\operatorname{vec}[L] = 0_{n^4\times 1}$, etc. □

Next we shall give two explicit parametrisations of the set $\varOmega(n, \mathbb{R})$, which in particular yield the dimension of the space of real Lyapunov operators. For this purpose we shall need the Jordan form $J_n$ of $\varPi_n$. The matrix $\varPi_n$ has two eigenvalues:

$\lambda_1 = 1$ with mupltiplicity $n_1 := n(n+1)/2$ and $\lambda_2 = -1$ with multiplicity $n_2 := n(n-1)/2$. Thus the Jordan form of $\Pi_n$ is

$$J_n = \Theta_n^{\mathrm{T}} \Pi_n \Theta_n = \mathrm{diag}(I_{n_1}, -I_{n_2}), \tag{22}$$

where the orthogonal matrix $\Theta_n \in \mathbb{R}^{n^2 \times n^2}$ may be obtained as follows. The permutation $L \mapsto \Pi_n L$ leaves $n$ rows of $L$ at their positions $(k-1)n+k$, $k = 1, \ldots, n$, and interchanges the positions of the rows in the remaining $n(n-1)/2$ pairs of rows. Hence there is a permutation matrix $\Theta_n'$, such that

$$(\Theta_n')^{\mathrm{T}} \Pi_n \Theta_n' = \mathrm{diag}\left( I_n, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Let

$$\Theta_n'' := \mathrm{diag}\left( I_n, \begin{bmatrix} \omega & -\omega \\ \omega & \omega \end{bmatrix}, \ldots, \begin{bmatrix} \omega & -\omega \\ \omega & \omega \end{bmatrix} \right), \quad \omega := 1/\sqrt{2}.$$

Then

$$\left( \Theta_n' \Theta_n'' \right)^{\mathrm{T}} \Pi_n \Theta_n' \Theta_n'' = \mathrm{diag}(I_n, 1, -1, \ldots, 1, -1).$$

Let $\Theta_2''' = I_4$ and, if $n > 2$, let $\Theta_n'''$ be the permutation matrix, corresponding to the permutation $n + 2l \leftrightarrow n^2 + 1 - 2l$, $l = 1, \ldots, (n-1)(n-2)/2$, leaving the other elements of $\{1, \ldots, n^2\}$ unchanged. Then

$$\Theta_n = \Theta_n' \Theta_n'' \Theta_n'''. \tag{23}$$

**Example 5.** For $n = 2$ the transformation of $\Pi_2$ into $J_2$ is done via

$$\Theta_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \omega & -\omega \\ 0 & 0 & \omega & \omega \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad J_2 = \Theta_2^{\mathrm{T}} \Pi_2 \Theta_2 = \mathrm{diag}(1, 1, 1, -1).$$

**Proposition 8.** *The subspace $\Omega(n, \mathbb{R})$ is parametrised as*

$$\begin{aligned} \Omega(n, \mathbb{R}) &= \mathscr{V}_n^{-1}(\mathbf{Her}(n^2, \mathbb{R})) \\ &= \left\{ \Theta_n \begin{bmatrix} L_{1,1} & 0 \\ 0 & L_{2,2} \end{bmatrix} \Theta_n^{\mathrm{T}} : L_{i,i} \in \mathbb{R}^{n_i \times n_i} \right\}. \end{aligned}$$

*In particular the* (*real*) *dimension of* **Lyap**$(n, \mathbb{R})$ *and* $\Omega(n, \mathbb{R})$ *is* $n_1^2 + n_2^2 = n^2(n^2+1)/2$.

**Proof.** The first parametrisation of $\Omega(n, \mathbb{R})$ follows immediately from Proposition 5(iv) and we see that the dimension of $\Omega(n, \mathbb{R})$ is that of $\mathbf{Her}(n^2, \mathbb{R})$, i.e., $n^2(n^2+1)/2$.

Consider the second parametrisation. The matrix equation $\Pi_n L = L \Pi_n$ for the matrix $L$ is equivalent to

$$J_n\widehat{L} = \widehat{L}J_n, \ \widehat{L} := \Theta_n^{\mathrm{T}} L \Theta_n. \tag{24}$$

The general solution of Eq. (24) is of the form $\widehat{L} = \mathrm{diag}(L_{1,1}, L_{2,2})$, where the matrices $L_{i,i} \in \mathbb{R}^{n_i \times n_i}$ are arbitrary, which completes the proof. □

**Example 6.** For $n = 2$ and $n = 3$ the sets $\Omega(2, \mathbb{R})$ and $\Omega(3, \mathbb{R})$ are 10- and 45-dimensional real spaces with patterns $\Lambda_2$ and $\Lambda_3$ of the free parameters as follows:

$$\Lambda_2 = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 4 & 6 & 5 & 7 \\ 8 & 9 & 9 & 10 \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 & 5 & 3 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 7 & 10 & 13 & 8 & 11 & 14 & 9 & 12 & 15 \\ 25 & 26 & 27 & 26 & 28 & 29 & 27 & 29 & 30 \\ 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 \\ 16 & 19 & 22 & 17 & 20 & 23 & 18 & 21 & 24 \\ 31 & 34 & 37 & 32 & 35 & 38 & 33 & 36 & 39 \\ 40 & 41 & 42 & 41 & 43 & 44 & 42 & 44 & 45 \end{bmatrix}.$$

If $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{R})$ is a general Sylvester operator, then according to Definition 3 we have

$$\|\mathscr{M}\|_{\mathrm{F}} := \sigma_{\max}(\mathscr{M}) := \sigma_1[\mathrm{Mat}(\mathscr{M})] = \max\{\|\mathscr{M}(X)\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1\}.$$

Similarly

$$\sigma_{\min}(\mathscr{M}) := \sigma_{n^2}[\mathrm{Mat}(\mathscr{M})] = \min\{\|\mathscr{M}(X)\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1\}$$

and if $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{R})$ is invertible, then $\|\mathscr{M}^{-1}\|_{\mathrm{F}} = 1/\sigma_{\min}(\mathscr{M})$.

For Lyapunov operators $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$, however, in addition to the standard maximum and minimum singular values $\sigma_{\max}(\mathscr{L})$ and $\sigma_{\min}(\mathscr{L})$, we may also define the maximum and minimum *Lyapunov singular values*

$$\|\mathscr{L}\|_{\widetilde{\mathrm{F}}} := \widetilde{\sigma}_{\max}(\mathscr{L}) := \max\{\|\mathscr{L}[X]\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1, \ X = X^{\mathrm{T}}\}$$

and

$$\widetilde{\sigma}_{\min}(\mathscr{L}) := \min\{\|\mathscr{L}[X]\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1, \ X = X^{\mathrm{T}}\}.$$

If $\mathscr{L}$ is invertible, then $\|\mathscr{L}^{-1}\|_{\widetilde{\mathrm{F}}} = 1/\widetilde{\sigma}_{\min}(\mathscr{L})$. Obviously

$$\sigma_{\min}(\mathscr{L}) \leqslant \widetilde{\sigma}_{\min}(\mathscr{L}) \leqslant \widetilde{\sigma}_{\max}(\mathscr{L}) \leqslant \sigma_{\max}(\mathscr{L}).$$

Each of these inequalities may be strict, i.e., the inequalities $\sigma_{\min}(\mathscr{L}) < \widetilde{\sigma}_{\min}(\mathscr{L})$ and $\widetilde{\sigma}_{\max}(\mathscr{L}) < \sigma_{\max}(\mathscr{L})$ are possible. Moreover, as we show below, the differences $\widetilde{\sigma}_{\min}(\mathscr{L}) - \sigma_{\min}(\mathscr{L})$ and $\sigma_{\max}(\mathscr{L}) - \widetilde{\sigma}_{\max}(\mathscr{L})$ may be arbitrarily large, see Example 8.

Let $A \in \mathbb{R}^{n\times n}$ and $a := \mathrm{vec}[A] \in \mathbb{R}^{n^2}$. The map $\mathrm{vec} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2}$ is an isomorphism and its inverse $\mathrm{vec}_n^{-1} : \mathbb{R}^{n^2} \to \mathbb{R}^{n\times n}$ is well defined. Using the notation $\mathrm{vec}_n^{-\mathrm{T}}(a) = (\mathrm{vec}_n^{-1}(a))^{\mathrm{T}}$ define the set

$$Z(n) := \left\{a \in \mathbb{R}^{n^2} : \mathrm{vec}_n^{-1}(a) = \mathrm{vec}_n^{-\mathrm{T}}(a)\right\},$$

corresponding to the symmetric matrices $A = \mathrm{vec}_n^{-1}(a)$, which is an $n(n+1)/2$-dimensional subspace of $\mathbb{R}^{n^2}$. We will show that

$$Z(n) = \mathrm{Rg}[I_{n^2} + \Pi_n] = \mathrm{Rg}[P_n],$$

where

$$P_n = [P_{n,ij}] \in \mathbb{R}^{n^2\times n(n+1)/2}, \quad i, j = 1, \ldots, n,$$

is an upper block-triangular matrix. The blocks $P_{n,ij} \in \mathbb{R}^{n\times j}$ are defined from

$$P_{n,ij} := \begin{cases} 0_{n\times j} & \text{if } i > j, \\ \begin{bmatrix} I_i \\ 0_{(n-i)\times i} \end{bmatrix} & \text{if } i = j, \\ E_{ji}(n, j) & \text{if } i < j. \end{cases}$$

If $L$ is the matrix representation of $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$, then we can rewrite the expression for $\widetilde{\sigma}_{\max}(\mathscr{L})$ in the equivalent form

$$\begin{aligned} \widetilde{\sigma}_{\max}(\mathscr{L}) &= \max\left\{\frac{\|La\|_2}{\|a\|_2} : 0 \neq a \in Z(n)\right\} \\ &= \max\left\{\frac{\|LP_nb\|_2}{\|P_nb\|_2} : 0 \neq b \in \mathbb{R}^{n(n+1)/2}\right\} \\ &= \|LQ_n\|_2 = \sigma_{\max}[LQ_n], \end{aligned}$$

where

$$Q_n := P_n(P_n^{\mathrm{T}}P_n)^{-1} = [Q_{n,ij}] \in \mathbb{R}^{n^2\times n(n+1)/2}, \quad i, j = 1, \ldots, n \tag{25}$$

is an upper block-triangular projector ($Q_n^{\mathrm{T}}Q_n = I_{n(n+1)/2}$). The blocks $Q_{n,ij} \in \mathbb{R}^{n\times j}$ are given by $Q_{n,ij} = 0$ if $i > j$, $Q_{n,11} = [1, 0, \ldots, 0]^{\mathrm{T}} \in \mathbb{R}^n$, $Q_{n,kk} = [\mathrm{diag}(\omega I_{k-1}, 1), 0]^{\mathrm{T}}$ and $Q_{n,ij} = \omega E_{ji}(n, j)$ if $i < j$, where $\omega := 1/\sqrt{2}$.

The matrices $P_n$ and $Q_n$ have the same sign-patterns, the only difference being that the non-zero elements of $P_n$ are equal to 1, while the non-zero elements of $Q_n$ are equal to 1 or $\omega$.

**Example 7.** The matrices $Q_2, Q_3, Q_4$ are

$$Q_2 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & \omega & 0 \\ \hline 0 & \omega & 0 \\ 0 & 0 & 1 \end{array}\right], \quad Q_3 = \left[\begin{array}{c|cc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ \hline 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ \hline 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$Q_4 = \left[\begin{array}{c|cc|ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ \hline 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ \hline 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Similarly, we have for the minimum Lyapunov singular value

$$\widetilde{\sigma}_{\min}(\mathscr{L}) = \sigma_{\min}[LQ_n].$$

**Definition 6.** The singular values of the matrix $LQ_n$ are called *Lyapunov singular values* of the Lyapunov operator $\mathscr{L}$ with associated matrix $L$. The set of Lyapunov singular values of $\mathscr{L}$ is denoted as

$$\widetilde{\sigma}(\mathscr{L}) := \sigma[LQ_n].$$

To compare the standard and Lyapunov maximum and minimum singular values, consider the following example.

**Example 8.** Let the operators $\mathscr{L}_1, \mathscr{L}_2 \in \mathbf{Lyap}(n, \mathbb{R})$ be determined from

$$\mathscr{L}_1[X] := E_{11}XE_{22} + E_{22}XE_{11} - E_{12}XE_{12} - E_{21}XE_{21},$$

$$\mathscr{L}_2[X] := X + \beta\mathscr{L}_1[X], \quad X \in \mathbb{R}^{2\times 2},$$

where $E_{ij} := E_{ij}(2,2)$ and $\beta > -1/2$. Setting $L_i := \mathrm{Mat}(\mathscr{L}_i)$ we have

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\beta & -\beta & 0 \\ 0 & -\beta & 1+\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\sigma_{\max}(\mathscr{L}_1) = 2$ and $L_1 Q_2 = 0_{4\times 3}$, the maximum singular value $\sigma_{\max}(\beta\mathscr{L}_1) = 2|\beta|$ of the operator $\beta\mathscr{L}_1$ may be arbitrarily larger than its maximum Lyapunov singular value $\widetilde{\sigma}_{\max}(\beta\mathscr{L}_1) = 0$. Furthermore we have $\sigma(\mathscr{L}_2) = \{2\beta+1, 1, 1, 1\}$ and since $L_2 Q_2 = Q_2$ we obtain $\widetilde{\sigma}(\mathscr{L}_2) = \{1, 1, 1\}$. Then for large $\beta$ the maximum singular value $\sigma_{\max}(\mathscr{L}_2) = 2\beta + 1$ of $\mathscr{L}_2$ is arbitrarily larger than its maximum Lyapunov singular value $\widetilde{\sigma}_{\max}(\mathscr{L}_2) = 1$. Finally, let $\beta = -1/2 + \varepsilon/2$, where $\varepsilon > 0$ is a small parameter. Then the minimum singular value $\sigma_{\min}(\mathscr{L}_2) = \varepsilon$ of $\mathscr{L}_2$ may be arbitrarily smaller than its minimum Lyapunov singular value, which is equal to 1.

The relationship between the sets of standard and Lyapunov singular values of a Lyapunov operator is revealed by the following assertion.

**Proposition 9.** *If $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$, then $\widetilde{\sigma}(\mathscr{L}) \subset \sigma(\mathscr{L})$.*

**Proof.** The set $\mathbf{Her}(n, \mathbb{R})$ is an invariant subspace of the operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{R})$. The orthogonal complement of that invariant subspace, the set $\mathbf{Sher}(n, \mathbb{R})$, is also an invariant subspace of $\mathscr{L}$. It follows that $\widetilde{\sigma}(\mathscr{L}) \subset \sigma(\mathscr{L})$. □

From application viewpoint it is important to define the class of Lyapunov operators $\mathscr{L}$ with Sylvester index $\mathrm{ind}_n(\mathscr{L}) \leqslant 2$ such that

$$\sigma_{\min}(\mathscr{L}) = \widetilde{\sigma}_{\min}(\mathscr{L}) \quad \text{and} \quad \sigma_{\max}(\mathscr{L}) = \widetilde{\sigma}_{\max}(\mathscr{L}). \tag{26}$$

As Example 8 shows, for $\mathrm{ind}_n(\mathscr{L}) \geqslant 4$ it is possible that $\sigma_{\min}(\mathscr{L}) < \widetilde{\sigma}_{\min}(\mathscr{L})$ and/or $\sigma_{\max}(\mathscr{L}) > \widetilde{\sigma}_{\max}(\mathscr{L})$. Using the results from [4] it may also be shown that for $n = 3$ and $\mathrm{ind}_3(\mathscr{L}) = 2$ relation (26) is not valid in general.

If (26) holds, then for Lyapunov operators that are most used in practice, e.g. for the descriptor continuous- and discrete-time operators $\mathscr{L}_{\mathrm{c}}$ and $\mathscr{L}_{\mathrm{d}}$, given by

$$\mathscr{L}_{\mathrm{c}}[X] = A^{\mathrm{T}}XE + E^{\mathrm{T}}XA, \;\; \mathscr{L}_{\mathrm{d}}[X] = A^{\mathrm{T}}XA - E^{\mathrm{T}}XE,$$

it is justified to use the minimum and maximum standard singular values since they are equal to the corresponding Lyapunov singular values. For general Lyapunov operators, however, it is better to use the Lyapunov singular values, since they produce sharper bounds.

Note finally that the converse of Proposition 9 is not true, i.e., the inclusion $\widetilde{\sigma}(\mathscr{M}) \subset \sigma(\mathscr{M})$ for some $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{R})$ does not imply $\mathscr{M} \in \mathbf{Lyap}(n, \mathbb{R})$, as is demonstrated in the following example.

**Example 9.** Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\widetilde{\sigma}(\mathscr{M}) = \sigma[MQ_2] = \{\sqrt{10}, 1, 1\} \subset \sigma[M] = \{\sqrt{10}, 1, 1, 0\},$$

but $M \notin \Omega(2, \mathbb{R})$ and hence the corresponding $\mathscr{M}$ is not a Lyapunov operator.

In Example 10, we consider the basic continuous-time and discrete-time Lyapunov operators from $\mathbf{Lyap}(2, \mathbb{R})$.

**Example 10.** Given the matrix $A \in \mathbb{R}^{2\times 2}$, the continuous-time operator $\mathscr{L}_{A,\mathrm{c}} \in \mathbf{Lyap}(2, \mathbb{R})$, defined by

$$\mathscr{L}_{A,\mathrm{c}}[X] := \mathscr{E}_2(A^{\mathrm{T}}, I_2, I_2, A) = A^{\mathrm{T}}X + XA, \quad X \in \mathbb{R}^{2\times 2}$$

is invertible if and only if $\mathrm{tr}[A] \neq 0$ and $\det[A] \neq 0$. Also, it is of index 1 if and only if $A$ is a scalar multiple of $I_2$, and of index 2 otherwise.

Given the matrix $A \in \mathbb{R}^{2\times 2}$, the discrete-time operator $\mathscr{L}_{A,\mathrm{d}} \in \mathbf{Lyap}(2, \mathbb{R})$, defined by

$$\mathscr{L}_{A,\mathrm{d}}[X] = \mathscr{E}_2(A^{\mathrm{T}}, A, I_n, -I_n) = A^{\mathrm{T}}XA - X, \quad X \in \mathbb{R}^{2\times 2}$$

is invertible if and only if $\det[A] \neq 1$ and $\mathrm{tr}[A] - \det[A] \neq 1$. It is of index 1 if and only if $A$ is a scalar multiple of $I_2$, and of index 2 otherwise.

The following example demonstrates that the continuous-time indexes of a Lyapunov operator and its inverse may be different.

**Example 11.** Let

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

where $0 \neq \lambda \in \mathbb{R}$. Then

$$\mathscr{L}_{A,\mathrm{c}}[X] = \begin{bmatrix} 2\lambda x_{1,1} & x_{1,1} + 2\lambda x_{1,2} \\ x_{1,1} + 2\lambda x_{2,1} & x_{2,1} + x_{1,2} + 2\lambda x_{2,2} \end{bmatrix},$$

$$X = [x_{i,j}] \in \mathbb{R}^{2\times 2},$$

$$\mathscr{L}^{-1}_{A,\mathrm{c}}[Y] = l \begin{bmatrix} y_{1,1} & y_{1,2} - l y_{1,1} \\ y_{2,1} - l y_{1,1} & 2l^2 y_{1,1} - l y_{2,1} - l y_{1,2} + y_{2,2} \end{bmatrix},$$

$$Y=[y_{i,j}] \in \mathbb{R}^{2\times 2},$$

where $l := 1/(2\lambda)$. Hence the matrix $L_{A,\mathrm{c}}$ of $\mathscr{L}_{A,\mathrm{c}}$ is

$$L_{A,\mathrm{c}} = \begin{bmatrix} 2\lambda & 0 & 0 & 0 \\ 1 & 2\lambda & 0 & 0 \\ 1 & 0 & 2\lambda & 0 \\ 0 & 1 & 1 & 2\lambda \end{bmatrix}.$$

The matrix

$$\mathscr{V}_2(L_{A,\mathrm{c}}) = \begin{bmatrix} 2\lambda & 1 & 0 & 2\lambda \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2\lambda & 0 \\ 2\lambda & 1 & 0 & 2\lambda \end{bmatrix}$$

has two eigenvalues $2\lambda \pm \sqrt{4\lambda^2+2}$ of opposite sign and two zero eigenvalues. Hence

$$\mathrm{clind}_2(\mathscr{L}_{A,\mathrm{c}}) = \mathrm{dlind}_2(\mathscr{L}_{A,\mathrm{c}}) = 2.$$

The matrix $\mathscr{V}_2(L_{A,\mathrm{c}}^{-1}) \in \mathbf{Her}(4, \mathbb{R})$ has two eigenvalues of the same sign and two zero eigenvalues which gives

$$\mathrm{dlind}_2(\mathscr{L}_{A,\mathrm{c}}^{-1}) = 2,\ \mathrm{clind}_2(\mathscr{L}_{A,\mathrm{c}}^{-1}) = 4.$$

If e.g. $\lambda > 0$ we have the following discrete-time representation of $\mathscr{L}_{A,\mathrm{c}}^{-1}$

$$\mathscr{L}_{A,\mathrm{c}}^{-1}[Y] = D_1 Y D_1^{\mathrm{T}} + D_2 Y D_2^{\mathrm{T}},$$

where

$$D_1 := \sqrt{l}\begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix}, \quad D_2 := l\sqrt{l}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

### *3.2. Complex Lyapunov operators*

The results, obtained for real Lyapunov operators, have their counterparts for complex Lyapunov operators, defined next.

An operator $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{C})$ may always be represented in the form (3), where $A_k, B_k \in \mathbb{C}^{n\times n}$. Definition 3 is directly applicable to such operators and Proposition 4 holds as well. Definition 4 is modified as follows.

**Definition 7.** The complex operator $\mathscr{L} \in \mathbf{Lin}(n, \mathbb{C})$ is said to be a *Lyapunov operator* if

$$(\mathscr{L}[X])^{\mathrm{H}} = \mathscr{L}[X^{\mathrm{H}}], \quad X \in \mathbb{C}^{n\times n}.$$

The set of complex Lyapunov operators is denoted by $\mathbf{Lyap}(n, \mathbb{C})$.

It follows from Definition 7 that

$$X = X^{\mathrm{H}} \Rightarrow \mathscr{L}[X] = (\mathscr{L}[X])^{\mathrm{H}},$$
$$X = -X^{\mathrm{H}} \Rightarrow \mathscr{L}[X] = -(\mathscr{L}[X])^{\mathrm{H}}$$

provided $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{C})$. Therefore $\mathbf{Her}(n, \mathbb{C})$ and $\mathbf{Sher}(n, \mathbb{C})$ are invariant sets for complex Lyapunov operators.

In the complex case, due to the non-linearity of the complex conjugation, the set $\mathbf{Lyap}(n, \mathbb{C}) \subset \mathbf{Lin}(n, \mathbb{C})$ of Lyapunov operators is not a subspace of $\mathbf{Lin}(n, \mathbb{C})$ and the set $\Omega(n, \mathbb{C}) \subset \mathbb{C}^{n^2\times n^2}$ is not a subspace of $\mathbb{C}^{n^2\times n^2}$ (these sets may become subspaces if we consider linear spaces of complex matrices with $\mathbb{R}$ as a field of scalars or if we pass to the representation $\mathbb{C}^{n^2\times n^2} \simeq \mathbb{R}^{2n^2\times 2n^2}$).

We have the following analogue of Proposition 5 in the complex case.

**Proposition 10.** *The following four statements are equivalent*:

(i) $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{C})$.

(ii) *There exists* $\mathscr{M} \in \mathbf{Lin}(n, \mathbb{C})$, *such that*

$$\mathscr{L}[X] = \mathscr{M}[X] + (\mathscr{M}[X^{\mathrm{H}}])^{\mathrm{H}}, \quad X \in \mathbb{C}^{n\times n},$$

*i.e.*,

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left(A_k X B_k + B_k^{\mathrm{H}} X A_k^{\mathrm{H}}\right)$$

*and*

$$L := \mathrm{Mat}(\mathscr{L}) = \sum_{k=1}^{r} \left(B_k^{\mathrm{T}} \otimes A_k + \overline{A}_k \otimes B_k^{\mathrm{H}}\right),$$

*where* $A_k, B_k \in \mathbb{C}^{n\times n}$ *are given matrices.*

(iii) $L \in \Omega(n, \mathbb{C})$, *where* $\Omega(n, \mathbb{C})$ *is the set of complex* $n^2 \times n^2$ *matrices* $L$, *satisfying the equation* $\Pi_n L = \overline{L}\Pi_n$.

(iv) *The matrix* $\mathbf{L} := \mathscr{V}_n(L)$ *is Hermitian.*

**Proof.** The proof is similar to that of Proposition 5. In particular we have the equation

$$\mathbf{AB} + (\mathbf{AB})^{\mathrm{H}} = \mathbf{L},$$

showing that $\mathbf{L}$ is Hermitian. □

If we represent $L \in \mathbb{C}^{n^2\times n^2}$ as $L = S + \jmath T$, where $S, T \in \mathbb{R}^{n^2\times n^2}$, then Proposition 10(iii) yields

$$\Pi_n S = S\Pi_n,\ \ \Pi_n T = -T\Pi_n. \tag{27}$$

Hence we come to the following analogues of Propositions 7 and 8.

**Proposition 11.** *The set $\Omega(n,\mathbb{C}) \subset \mathbb{C}^{n^2\times n^2}$ of matrix representations of complex Lyapunov operators is isomorphic to the subspace*

$$\begin{aligned}&\operatorname{Ker}\left[\operatorname{diag}\left(I_{n^2}\otimes\Pi_n-\Pi_n\otimes I_{n^2},\, I_{n^2}\otimes\Pi_n+\Pi_n\otimes I_{n^2}\right)\right]\\ &\quad=\operatorname{Ker}\left[\operatorname{diag}\left(\Pi_n\otimes\Pi_n-I_{n^4},\,\Pi_n\otimes\Pi_n+I_{n^4}\right)\right]\subset\mathbb{R}^{2n^4}.\end{aligned}$$

**Proof.** The proof follows directly from (27). □

Using the Jordan form (22) of $\Pi_n$ and the matrix $\Theta_n$ from (23) we can parametrize the set $\Omega(n,\mathbb{C})$ and determine its real dimension according to the following proposition.

**Proposition 12.** *The set $\Omega(n,\mathbb{C})$ is parametrised as*

$$\begin{aligned}\Omega(n,\mathbb{C})&=\mathscr{V}_n^{-1}(\mathbf{Her}(n^2,\mathbb{C}))\\ &=\left\{\Theta_n\begin{bmatrix}L_{1,1} & \jmath L_{1,2}\\ \jmath L_{2,1} & L_{2,2}\end{bmatrix}\Theta_n^{\mathrm{T}} : L_{i,j}\in\mathbb{R}^{n_i\times n_j}\right\}.\end{aligned}$$

*In particular the real dimension of* **Lyap**$(n,\mathbb{C})$ *and* $\Omega(n,\mathbb{C})$ *is* $n^4$.

**Proof.** The first representation follows from Proposition 10(iv). The second one is based on Eq. (27) for the matrices $S$ and $T$. Using the Jordan form $J_n$ of $\Pi_n$ we obtain the equivalent equations

$$J_n\widehat{S}=\widehat{S}J_n,\quad \widehat{S}:=\Theta_n^{\mathrm{T}}S\Theta_n,\qquad J_n\widehat{T}=-\widehat{T}J_n,\quad \widehat{T}:=\Theta_n^{\mathrm{T}}T\Theta_n. \tag{28}$$

The general solution of (28) is of the form

$$\widehat{S}=\begin{bmatrix}L_{1,1} & 0\\ 0 & L_{2,2}\end{bmatrix},\quad \widehat{T}=\begin{bmatrix}0 & L_{1,2}\\ L_{2,1} & 0\end{bmatrix},$$

where the matrices $L_{i,j}\in\mathbb{R}^{n_i\times n_j}$ are arbitrary. The proof is complete. □

Similarly to the real case, a complex Lyapunov operator $\mathscr{L}\in$ **Lyap**$(n,\mathbb{C})$ admits also the Hermitian representation

$$\mathscr{L}[X]=\sum_{j=1}^{\ell_{\mathrm{d}}}\varepsilon_j D_j X D_j{}^{\mathrm{H}},$$

where $\varepsilon_j=\pm1$ and $D_j\in\mathbb{C}^{n\times n}$. Accordingly, the concepts from Definition 5 are easily extended to the case of complex Lyapunov operators. In particular we see that Proposition 6 holds true also in the complex case.

The maximum and minimum Lyapunov singular values of the operator $\mathscr{L}\in$ **Lyap** $(n,\mathbb{C})$are defined as

$$\begin{aligned}\widetilde{\sigma}_{\max}(\mathscr{L}) &:= \max\{\|\mathscr{L}[X]\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1,\ X = X^{\mathrm{H}}\},\\ \widetilde{\sigma}_{\min}(\mathscr{L}) &:= \min\{\|\mathscr{L}[X]\|_{\mathrm{F}} : \|X\|_{\mathrm{F}} = 1,\ X = X^{\mathrm{H}}\},\end{aligned} \tag{29}$$

respectively.

The Lyapunov singular values of a complex Lyapunov operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{C})$ with matrix

$$L = S + \jmath T, \quad S \in \mathbf{Her}(n, \mathbb{R}), \quad T \in \mathbf{Sher}(n, \mathbb{R})$$

are defined as follows. Let

$$L^{\mathbb{R}} := \begin{bmatrix} S & -T \\ T & S \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

be the realification of $L$. Let $X = Y + \jmath Z$; $Y, Z \in \mathbb{R}^{n\times n}$. Then the restriction $X = X^{\mathrm{H}}$ in (29) means that $Y$ is symmetric and $Z$ is skew-symmetric, i.e.,

$$y := \mathrm{vec}[Y] \in \mathrm{Rg}[I_{n^2} - \Pi_n], \quad z := \mathrm{vec}[Z] \in \mathrm{Rg}[I_{n^2} + \Pi_n].$$

Furthermore, we have

$$\|\mathscr{L}[X]\|_{\mathrm{F}} = \left\| L^{\mathbb{R}} \begin{bmatrix} y \\ z \end{bmatrix} \right\|_2 .$$

Therefore, as in the real case, we obtain

$$\widetilde{\sigma}_{\max}(\mathscr{L}) = \|\Psi_n[L]\|_2 = \sigma_1[\Psi_n[L]], \quad \widetilde{\sigma}_{\min}(\mathscr{L}) = \sigma_{n^2}[\Psi_n[L]],$$

where the matrix $\Psi_n[L]$ is defined from

$$\Psi_n[L] := L^{\mathbb{R}}\mathrm{diag}(Q_n, R_n) = \begin{bmatrix} SQ_n & -TR_n \\ TQ_n & SR_n \end{bmatrix} \in \mathbb{R}^{2n^2\times n}.$$

Here the matrix $R_n \in \mathbb{R}^{n^2\times n(n-1)/2}$ is obtained from $Q_n$ (see (25) and Example 7) by deleting the columns containing 1's (which are numbered as $k(k+1)/2$, $k = 1, \ldots, n$) and by changing the sign of each second element $\omega$ in each column of the reduced matrix. Formally this procedure is described as follows. Let

$$\Delta_n = [\Delta_n]_{i,j} := [\delta_{i(i+1)/2,j}] \in \mathbb{R}^{n(n+1)/2\times n(n-1)/2},$$

where $\delta_{i,j}$ is the Kronecker delta, and

$$\mathscr{J} := \{(kn + l, k(k-1)/2 + l) : k = 1, \ldots, n-1,\ l = 1, \ldots, k\}.$$

Then the elements $[R_n]_{i,j}$ of the matrix $R_n$ are given by

$$[R_n]_{i,j} = \begin{cases} [Q_n\Delta_n]_{i,j} & \text{if } (i, j) \notin \mathscr{J}, \\ -[Q_n\Delta_n]_{i,j} & \text{if } (i, j) \in \mathscr{J}. \end{cases}$$

**Example 12.** The matrices $R_2, R_3, R_4$ are

$$R_2 = \left[\begin{array}{c} 0 \\ \omega \\ \hline -\omega \\ 0 \end{array}\right], \quad R_3 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \omega & 0 & 0 \\ 0 & -\omega & 0 \\ \hline -\omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega \\ \hline 0 & -\omega & 0 \\ 0 & 0 & -\omega \\ 0 & 0 & 0 \end{array}\right],$$

$$R_4 = \left[\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ \hline -\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ \hline 0 & -\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ \hline 0 & 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

**Definition 8.** The *Lyapunov singular values* of the complex Lyapunov operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{C})$ with associated matrix $L \in \Omega(n, \mathbb{C})$ are the singular values of the matrix $\Psi_n[L]$:

$$\widetilde{\sigma}(\mathscr{L}) := \sigma[\Psi_n[L]].$$

A similar statement $\widetilde{\sigma}(\mathscr{L}) \subset \sigma(\mathscr{L})$ as in the real case can be stated for complex Lyapunov operators $\mathscr{L}$.

Consider now some problems concerning the inversion of Lyapunov operators. The operator $\mathscr{L} \in \mathbf{Lyap}(n, \mathbb{F})$ is invertible if and only if its matrix $L$ is non-singular, and $\mathrm{Mat}(\mathscr{L}^{-1}) = L^{-1}$. In addition, the inverse of a Lyapunov operator is again a Lyapunov $\mathscr{L}$ operator since for $L \in \mathbf{GL}(n^2, \mathbb{F})$ the equations $\Pi_n L = \overline{L}\Pi_n$ and $\Pi_n L^{-1} = \overline{L}^{-1}\Pi_n$ are equivalent. Conditions for invertibility of general real and complex Lyapunov operators are given in [13].

Note finally that the continuous-time Lyapunov indexes of a Lyapunov operator and its inverse may differ, see Example 11.

### 3.3. Lyapunov-like operators

In this section we consider six more classes of Lyapunov operators and present their parametrisations and dimensions in particular. The proofs are similar to these from Section 3.2 and are omitted.

#### 3.3.1. Skew-Lyapunov operators

1. *Real case*. *Real skew-Lyapunov operators* $\mathscr{L}$ from $\mathbf{Lin}(n, \mathbb{R})$ are defined from

$$(\mathscr{L}[X])^{\mathrm{T}} = -\mathscr{L}[X^{\mathrm{T}}], \quad X \in \mathbb{R}^{n\times n}$$

and may be represented as

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left(A_k X B_k - B_k^{\mathrm{T}} X A_k^{\mathrm{T}}\right), \quad A_k, B_k \in \mathbb{R}^{n\times n}. \tag{30}$$

The matrix $L \in \mathbb{R}^{n^2\times n^2}$ of a skew-Lyapunov operator satisfies $\Pi_n L = -L\Pi_n$ and has the form

$$L = \Theta_n \begin{bmatrix} 0 & L_{1,2} \\ L_{2,1} & 0 \end{bmatrix} \Theta_n^{\mathrm{T}},$$

where the matrices $L_{i,j} \in \mathbb{R}^{n_i\times n_j}$ are arbitrary. Hence the space of real skew-Lyapunov operators is of dimension $2n_1n_2 = n^2(n^2-1)/2$. Since for real skew-Lyapunov operators we have

$$\mathbf{AB} - (\mathbf{AB})^{\mathrm{T}} = \mathbf{L},$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ are defined in (15), then the matrix $\mathbf{L} := \mathscr{V}_n(L)$ here is skew-symmetric.

2. *Complex case*. *Complex skew-Lyapunov operators* $\mathscr{L}$ from $\mathbf{Lin}(n, \mathbb{C})$ are defined by the relation

$$(\mathscr{L}[X])^{\mathrm{H}} = -\mathscr{L}[X^{\mathrm{H}}], \quad X \in \mathbb{C}^{n\times n}$$

and may be represented as

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left(A_k X B_k - B_k{}^{\mathrm{H}} X A_k{}^{\mathrm{H}}\right), \quad A_k, B_k \in \mathbb{C}^{n\times n}. \tag{31}$$

The matrix $L = S + \jmath T \in \mathbb{R}^{n^2\times n^2}$; $S, T \in \mathbb{R}^{n^2\times n^2}$, of a complex skew-Lyapunov operator $\mathscr{L}$ satisfies the equation $\Pi_n L = -\overline{L}\Pi_n$ and hence $\Pi_n S = -S\Pi_n$, $\Pi_n T = T\Pi_n$. Thus

$$L = \Theta_n \begin{bmatrix} \jmath L_{1,1} & L_{1,2} \\ L_{2,1} & \jmath L_{2,2} \end{bmatrix} \Theta_n^{\mathrm{T}},$$

where the matrices $L_{i,j} \in \mathbb{R}^{n_i\times n_j}$ are arbitrary. Hence the space of complex skew-Lyapunov operators is of real dimension $n^4$. The matrix $\mathbf{L} := \mathscr{V}_n(L)$ for a complex skew-Lyapunov operator $\mathscr{L}$ is skew-Hermitian since

$$\mathbf{AB} - (\mathbf{AB})^{\mathrm{H}} = \mathbf{L}.$$

3. *Skew-Lyapunov index of skew-Lyapunov operators.* The *skew-Lyapunov index* of a skew-Lyapunov operator is defined as the minimum number of terms in the representations (30) or (31) and may be determined as follows.

Consider the equation $\mathbf{C} - \mathbf{C}^* = \mathbf{L}$ in $\mathbf{C} := \mathbf{AB}$ for a skew-Lyapunov operator. The matrix $\mathbf{L}$ is congruent to the matrix

$$\operatorname{diag}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, 0_{(n^2-2r)\times(n^2-2r)}\right)$$

with $r$ blocks of size $2 \times 2$ on the diagonal in the real case, and to the matrix

$$\operatorname{diag}\left(\imath I_\alpha, -\imath I_\beta, 0_{(n^2-\gamma)\times(n^2-\gamma)}\right)$$

in the complex case, where $\gamma := \alpha + \beta = \operatorname{rank}[\mathbf{L}]$. Hence the minimum achievable rank of $\mathbf{C}$ is the rank of $\mathbf{L}$. Thus the skew-Lyapunov index of the skew-Lyapunov operator $\mathscr{L} \in \mathbf{Lin}(n, \mathbb{F})$ with matrix $L$ is equal to the rank of the matrix $\mathbf{L} := \mathscr{V}_n(L)$.

### 3.3.2. Associated Lyapunov operators

Associated Lyapunov and Riccati equations have been considered in [15] in the real case and in [14] in the complex case. Below we present the parametrisations of associated Lyapunov operators.

1. *Real case*. *Real associated Lyapunov operators* $\mathscr{L}$ from $\mathbf{Lin}(n, \mathbb{R})$ are defined from

$$(\mathscr{L}[X])^{\mathrm{T}} = \mathscr{L}[X], \quad X \in \mathbb{R}^{n\times n},$$

and are given by

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left(A_k X B_k + B_k^{\mathrm{T}} X^{\mathrm{T}} A_k^{\mathrm{T}}\right), \quad A_k, B_k \in \mathbb{R}^{n\times n}.$$

The matrix

$$L = \sum_{k=1}^{r} \left(B_k^{\mathrm{T}} \otimes A_k + A_k \otimes B_k^{\mathrm{T}} \varPi_n\right) \in \mathbb{R}^{n^2\times n^2}$$

of the associated Lyapunov operator $\mathscr{L}$ satisfies $\varPi_n L = L$ and has the form

$$L = \varTheta_n \begin{bmatrix} L_1 \\ 0_{n_2\times n^2} \end{bmatrix} \varTheta_n^{\mathrm{T}},$$

where the matrix $L_1 \in \mathbb{R}^{n_1\times n^2}$ is arbitrary. Hence the space of real associated Lyapunov operators is of dimension $n^3(n+1)/2$. It may be shown that $\mathbf{L}_{i,j} = \mathbf{L}_{i,j}^{\mathrm{T}}$, where $\mathbf{L}_{i,j}$ are the $n \times n$ blocks in the partition $\mathbf{L} = [\mathbf{L}_{i,j}]$ of the matrix $\mathbf{L} := \mathscr{V}_n[L]$.

2. *Complex case*. *Complex associated Lyapunov operators* $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ are defined by

$$(\mathscr{L}[X])^{\mathrm{H}} = \mathscr{L}[X], \quad X \in \mathbb{C}^{n\times n}$$

and may be represented as

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left( A_k X B_k + {B_k}^{\mathrm{H}} X^{\mathrm{H}} {A_k}^{\mathrm{H}} \right), \quad A_k, B_k \in \mathbb{C}^{n\times n}.$$

Complex associated Lyapunov operators are not linear, but pseudo-linear operators, see [14]. For pseudo-linear operators $\mathscr{L} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ we have

$$\mathscr{L}[X] = \mathscr{M}_1[X] + \mathscr{M}_2[X^{\mathrm{H}}], \quad \mathscr{M}_i \in \mathbf{Lin}(n, \mathbb{C})$$

and

$$\mathrm{vec}[\mathscr{L}[X]] = M_1 \mathrm{vec}[X] + M_2 \Pi_n \mathrm{vec}[\overline{X}], \quad M_i := \mathrm{Mat}(\mathscr{M}_i).$$

Thus the set of these pseudo-linear operators is of complex dimension $2n^4$.

After some calculations we see that for a complex associated Lyapunov operator it is fulfilled $M_2 = \overline{M}_1$, i.e.,

$$\mathrm{vec}[\mathscr{L}[X]] = \Lambda \mathrm{vec}[X] + \overline{\Lambda} \mathrm{vec}[\overline{X}], \;\; \Lambda \in \mathbb{C}^{n\times n}.$$

Hence the set of complex associated Lyapunov operators is of complex dimension $n^4$.

The values of an associated Lyapunov operator are symmetric matrices in the real case and Hermitian matrices in the complex case. Hence these operators are not surjective if considered as mappings $\mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$. Hence one should consider associated Lyapunov operators as mappings $\mathbb{F}^{n\times n} \to \mathbf{Her}(n, \mathbb{F})$.

### *3.3.3. Associated skew-Lyapunov operators*

1. *Real case*. *Real associated skew-Lyapunov operators* $\mathscr{L}$ from $\mathbf{Lin}(n, \mathbb{R})$ are defined from

$$(\mathscr{L}[X])^{\mathrm{T}} = -\mathscr{L}[X], \quad X \in \mathbb{R}^{n\times n}$$

and may be represented as

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left( A_k X B_k - B_k^{\mathrm{T}} X^{\mathrm{T}} A_k^{\mathrm{T}} \right), \quad A_k, B_k \in \mathbb{R}^{n\times n}.$$

The matrix

$$L = \sum_{k=1}^{r} \left( B_k^{\mathrm{T}} \otimes A_k - \left( A_k \otimes B_k^{\mathrm{T}} \Pi_n \right) \right) \in \mathbb{R}^{n^2\times n^2}$$

of an associated skew-Lyapunov operator satisfies $\Pi_n L = -L$ and has the form

$$L = \Theta_n \begin{bmatrix} 0_{n_1\times n^2} \\ L_2 \end{bmatrix} \Theta_n^{\mathrm{T}},$$

where the matrix $L_2 \in \mathbb{R}^{n_2\times n^2}$ is arbitrary. Hence the space of real associated skew-Lyapunov operators is of dimension $n_2 n^2 = n^3(n-1)/2$.

It may be proved that $\mathbf{L}_{i,j} = -\mathbf{L}_{i,j}^{\mathrm{T}}$, where $\mathbf{L}_{i,j}$ are the $n \times n$ blocks in the partition of the matrix $\mathbf{L} := \mathscr{V}_n[L]$.

2. *Complex case*. *Complex associated skew-Lyapunov operators* $\mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ are determined by

$$(\mathscr{L}[X])^{\mathrm{H}} = -\mathscr{L}[X], \quad X \in \mathbb{C}^{n\times n}$$

and may be represented as

$$\mathscr{L}[X] = \sum_{k=1}^{r} \left( A_k X B_k - B_k{}^{\mathrm{H}} X^{\mathrm{H}} A_k{}^{\mathrm{H}} \right), \quad A_k, B_k \in \mathbb{C}^{n\times n}.$$

These operators are pseudo-linear and satisfy

$$\mathrm{vec}[\mathscr{L}[X]] = \Lambda \mathrm{vec}[X] - \overline{\Lambda} \mathrm{vec}[\overline{X}], \quad \Lambda \in \mathbb{C}^{n\times n}.$$

Thus the set of complex associated skew-Lyapunov operators is of complex dimension $n^4$.

The values of associated skew-Lyapunov operator are skew-symmetric matrices (in the real case) or skew-Hermitian matrices (in the complex case) and these operators are not surjective if considered as mappings $\mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$. Hence one may consider associated skew-Lyapunov operators as mappings $\mathbb{C}^{n\times n} \to \mathbf{Sher}(n, \mathbb{F})$.

### *3.4. Bilinear and quadratic matrix operators*

*Real bilinear matrix operators* $\mathscr{B} : \mathbb{R}^{m'\times n'} \times \mathbb{R}^{m''\times n''} \to \mathbb{R}^{p\times q}$ are defined from

$$\mathscr{B}[X, Y] = \sum_{k=1}^{r} A_k X B_k Y C_k, \quad A_k \in \mathbb{R}^{p\times m'}, \ \ B_k \in \mathbb{R}^{n'\times m''}, \ \ C_k \in \mathbb{R}^{n''\times q}.$$

For $m' = m'' = m$ and $n' = n'' = n$ we have the *real quadratic operator* $\mathscr{Q} : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$, determined from

$$\mathscr{Q}[X] = \sum_{k=1}^{r} A_k X B_k X C_k, \quad A_k \in \mathbb{R}^{p\times m}, \ \ B_k \in \mathbb{R}^{n\times m}, \ \ C_k \in \mathbb{R}^{n\times q}.$$

*Real symmetric quadratic* (or *real Riccati*) *operators* are given by $(\mathscr{Q}[X])^{\mathrm{T}} = Q[X^{\mathrm{T}}]$, i.e.,

$$\mathscr{Q}[X] = \sum_{k=1}^{r} \left( A_k X B_k X C_k + C_k^{\mathrm{T}} X B_k^{\mathrm{T}} X A_k^{\mathrm{T}} \right).$$

As for linear operators, we may define the *index* of a bilinear and a quadratic operator as well as the *Riccati index* of a Riccati operator.

Similar problems may be posed for complex bilinear, quadratic and Riccati operators.

At present little is known for the indexes of such operators. However, the index of the quadratic operator $\mathscr{Q} : \mathbb{F}^{m\times n} \to \mathbb{F}^{p\times q}$ may easily be determined when $n = 1$.

Indeed, let $\mathscr{Q}$ be given by $mp$ quadratic forms $\omega_{i,j}[X] := X^* Q_{i,j} X$; $i = 1, \ldots, m$, $j = 1, \ldots, p$, where $Q_{i,j} \in \mathbf{Her}(n, \mathbb{F})$. Let the $n \times pmn$ matrix $Q$ be formed by the columns of the matrices $Q_{i,j}$. Then the index of $\mathscr{Q}$ is the number of non-zero columns of the matrix $Q$, and the matrices $A_k, B_k, C_k$ may easily be constructed from the data $Q_{i,j}$. The next example gives the idea how to do that.

**Example 13.** Let $p = m = 3$, $n = q = 1$. Then the real quadratic operator $\mathscr{Q} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by three matrices $Q^{(k)} = [q^{(k)}_{i,j}] \in \mathbf{Her}(3, \mathbb{R})$,

$$\mathscr{Q}[X] = \left[ X^{\mathrm{T}} Q^{(1)} X, X^{\mathrm{T}} Q^{(2)} X, X^{\mathrm{T}} Q^{(3)} X \right]^{\mathrm{T}}.$$

Suppose that the matrix $Q := [Q^{(1)}, Q^{(2)}, Q^{(3)}] \in \mathbb{R}^{3\times 9}$ has no zero columns. Then $\mathscr{Q}$ is of index 9 and may be represented as

$$\mathscr{Q}[X] = \sum_{i,j=1}^{3,3} E_{i,j}(3,3) X B_{i,j} X,$$

where

$$B_{i,1} := [q^{(i)}_{1,1}, 2q^{(i)}_{1,2}, 2q^{(i)}_{1,3}], \quad B_{i,2} := [0, q^{(i)}_{2,2}, 2q^{(i)}_{2,3}], \quad B_{i,3} := [0, 0, q^{(i)}_{3,3}].$$

## 4. Computational aspects

The proofs of Propositions 4 and 6 are constructive since they describe procedures to compute the matrices $A_k$, $B_k$, $D_j$ in the representations of a Sylvester operator $\mathscr{M}$ with matrix $M$ or a Lyapunov operator $\mathscr{L}$ with matrix $L$. For this purpose the decompositions

$$\mathbf{M} := \mathscr{V}_{p,m}[M] = U \operatorname{diag}\left(I_r, 0_{(pm-r)\times(nq-r)}\right) V^{-1}$$

and

$$\mathbf{L} := \mathscr{V}_n[L] = P \operatorname{diag}\left(2I_\alpha, -2I_\beta, 0_{(n^2-\gamma)\times(n^2-\gamma)}\right) P^*$$

have been used, where $U \in \mathbf{GL}(pm, \mathbb{F})$, $V \in \mathbf{GL}(nq, \mathbb{F})$ and $P \in \mathbf{GL}(n^2, \mathbb{F})$. However, from computational point of view it is preferable to use unitary (or orthogonal in the real case) instead of general linear transformations. Choosing $U$, $V$ and $P$ as unitary or orthogonal matrices we have

$$\mathbf{M} = U \operatorname{diag}\left(\sigma_1[\mathbf{M}], \ldots, \sigma_r[\mathbf{M}], 0_{(pm-r)\times(nq-r)}\right) V^*$$

and

$$\mathbf{L} = P \operatorname{diag}\left(\Lambda_\alpha, -\Lambda_\beta, 0_{(n^2-\gamma)\times(n^2-\gamma)}\right) P^*,$$

where $\Lambda_\alpha$ and $-\Lambda_\beta$ are diagonal matrices, containing the positive and negative eigenvalues of $\mathbf{L}$, respectively. In this way, given the matrices $\mathbf{M}$ or $\mathbf{L}$, the computation of the matrices $A_k$, $B_k$, $D_j$ in the representations of the operators $\mathscr{M}$ and $\mathscr{L}$ may be done in a numerically stable way.

So far we have made an analysis of general Sylvester and Lyapunov operators. In the following section, we discuss the application of these results to the sensitivity and a posteriori error analysis of Lyapunov equations.

## 5. Application to the sensitivity and error analysis of Lyapunov equations

Consider the Hermitian Lyapunov equation

$$\mathscr{L}[X] = Q, \quad Q^{\mathrm{H}} = Q \neq 0 \tag{32}$$

with an invertible Lyapunov operator $\mathscr{L}$. The minimum symmetric singular value $\widetilde{\sigma}_{\min}(\mathscr{L})$ of $\mathscr{L}$ is a relevant measure for the sensitivity of the Lyapunov equation (32) relative to perturbations in the coefficient matrices of $\mathscr{L}$ and Hermitian perturbations $\varDelta Q = \varDelta Q^{\mathrm{H}}$ in the matrix $Q$.

Denote by $X_0 = X_0^{\mathrm{H}} = \mathscr{L}^{-1}[Q]$ the solution of (32) and let $X = X_0 + \varDelta X$ be the solution to the perturbed Lyapunov equation $\mathscr{L}(X) = Q + \varDelta Q$. We have $\varDelta X = \mathscr{L}^{-1}[\varDelta Q]$ and hence

$$\|\varDelta X\|_{\mathrm{F}} \leqslant \|\mathscr{L}^{-1}\|_{\widetilde{\mathrm{F}}} \|\varDelta Q\|_{\mathrm{F}} = \frac{1}{\widetilde{\sigma}_{\min}(\mathscr{L})} \|\varDelta Q\|_{\mathrm{F}}.$$

In terms of relative perturbations it is fulfilled that

$$\delta_X \leqslant \widetilde{\kappa}\delta_Q, \quad \widetilde{\kappa} := \frac{1}{\widetilde{\sigma}_{\min}(\mathscr{L})} \frac{\|Q\|_{\mathrm{F}}}{\|P\|_{\mathrm{F}}},$$

where $\delta_Z := \|\varDelta Z\|_{\mathrm{F}}/\|Z\|_{\mathrm{F}}$ and $\widetilde{\kappa}$ is the *relative condition number* of the Lyapunov equation (32) with respect to Hermitian perturbations in $Q$. Note that usually $Q = D^{\mathrm{H}}D$ and when the matrix $D$ is perturbed then the perturbation $\varDelta Q = \varDelta D^{\mathrm{H}}D + D^{\mathrm{H}}\varDelta D + \varDelta D^{\mathrm{H}}\varDelta D$ in $Q$ is Hermitian.

Most of the perturbation bounds in the literature [6,9] are based on $\sigma_{\min}(\mathscr{L})$ instead on $\widetilde{\sigma}_{\min}(\mathscr{L})$, e.g. the condition number is taken as $\kappa := \|Q\|_{\mathrm{F}}/(\|P\|_{\mathrm{F}}\sigma_{\min}(\mathscr{L}))$. Since $\kappa \geqslant \widetilde{\kappa}$ may be much larger than $\widetilde{\kappa}$, it is clear that in case of Hermitian perturbations one should use the relevant sensitivity estimates, based on symmetric singular values instead on standard singular values of Lyapunov operators. At the same time sensitivity estimates, based on the standard singular values, should be used in case of non-Hermitian perturbations.

Consider now the a posteriori error analysis of Eq. (32). Suppose that $\widehat{X}$ is an approximate solution of Eq. (32). For example this may be the solution, produced by a numerical method in finite arithmetics (e.g. in floating-point computing environment). Then it is important to have a sharp computable bound on the actual relative error

$$\delta_{\widehat{X}} := \frac{\|\widehat{X} - X_0\|_{\mathrm{F}}}{\|X_0\|_{\mathrm{F}}}.$$

Such a tight bound may be derived using the symmetric singular values of $\mathscr{L}$ and in particular the symmetric relative condition number of $\mathscr{L}$, defined below.

Denote by $\widehat{Q} := \mathscr{L}[\widehat{X}]$ the residual, corresponding to the approximate solution $\widehat{X}$. We have $\mathscr{L}[\widehat{X} - X_0] = \widehat{Q} - Q$, which gives $\widehat{X} - X_0 = \mathscr{L}^{-1}[\widehat{Q} - Q]$ and

$$\|\widehat{X} - X_0\|_{\mathrm{F}} \leqslant \frac{\|\widehat{Q} - Q\|_{\mathrm{F}}}{\widetilde{\sigma}_{\min}(\mathscr{L})}. \tag{33}$$

Since $\|Q\|_{\mathrm{F}} \leqslant \widetilde{\sigma}_{\max}(\mathscr{L})\|X_0\|_{\mathrm{F}}$ it is fulfilled that

$$\frac{1}{\|X_0\|_{\mathrm{F}}} \leqslant \frac{\widetilde{\sigma}_{\max}(\mathscr{L})}{\|Q\|_{\mathrm{F}}}. \tag{34}$$

Combining (33) and (34) we get the desired estimate

$$\delta_{\widehat{X}} \leqslant \widetilde{\mathrm{cond}}_2(\mathscr{L}) \frac{\|\widehat{Q} - Q\|_{\mathrm{F}}}{\|Q\|_{\mathrm{F}}},$$

where

$$\widetilde{\mathrm{cond}}_2(\mathscr{L}) := \frac{\widetilde{\sigma}_{\max}(\mathscr{L})}{\widetilde{\sigma}_{\min}(\mathscr{L})}$$

is the *symmetric relative condition number* of $\mathscr{L}$ with respect to inversion. This condition number may be used also for a posteriori error analysis of approximate solutions to symmetric matrix Riccati equations.

## 6. Conclusions and unsolved problems

The paper presents some new concepts and properties related to general Sylvester and Lyapunov operators and some of their applications. In particular the Sylvester index of a linear matrix operator is defined (as the minimum number of elementary Sylvester operators, necessary to determine the initial operator) and explicitely computed. Full characterizations of the spaces of real and complex Lyapunov operators are also given. The calculation of the Sylvester index is based on the possibility to solve exactly a bilinear matrix equation of the type $\mathbf{AB} = \mathbf{M}$ in the unknown matrices $\mathbf{A}$ and $\mathbf{B}$. Given a representation of a linear operator as a sum of elementary operators, it is easy to find its associated matrix. The inverse problem, namely to determine the representation of an operator as a sum of elementary operators on the basis of its associated matrix, is more difficult. Solving the above bilinear matrix equation one gets also the representation of the operator as a condensed sum of elementary operators, depending on the associated matrix. An interesting and yet unsolved problem is to approximate a given linear matrix operator by a sum of elementary operators, whose number is less than its Sylvester index. A possible approach is to solve the non-linear least squares problem $\|\mathbf{AB} - \mathbf{M}\| \to \min$.

For Lyapunov operator special symmetric representations are derived and the corresponding Lyapunov indexes are defined in addition to the Sylvester index. Similar problems arise for skew-Lyapunov and associated skew-Lyapunov operators.

The same concepts may be introduced for bilinear, quadratic and Riccati matrix operators, as well as for polynomial operators of higher order in both the real and complex cases. In this area there are practically no results.

## Acknowledgement

The authors would like to thank the anonymous referee for his very helpful comments and for pointing out the elegant proof of Proposition 9.

## References

[1] J.B. Barlow, M.M. Monahemi, D.P. O'Leary, Constrained matrix Sylvester equations, SIAM J. Matrix Anal. Appl. 13 (1992) 1–9.
[2] S. Barnett, Introduction to Mathematical Control Theory, Oxford University Press, Oxford, 1975.
[3] S. Barnett, C. Storey, Matrix Methods in Stability Theory, Nelson, London, 1970.
[4] R. Byers, S. Nash, On the singular "vectors" of the Lyapunov operator, SIAM J. Algebraic Discrete Methods 8 (1987) 59–66.
[5] D. Carlson, R. Loewy, On ranges of Lyapunov transformations, Linear Algebra Appl. 8 (1974) 237–248.
[6] P.M. Gahinet, A.J. Laub, C.S. Kenney, G.A. Hewer, Sensitivity of the stable discrete-time Lyapunov equation, IEEE Trans. Automat. Control 35 (1990) 1209–1217.
[7] Z. Gajić, M.T.J. Qureshi, Lyapunov Matrix Equation in System Stability and Control, Academic Press, San Diego, CA, 1995.
[8] W. Givens, Elementary divisors and some properties of the Lyapunov mapping $X \to AX + XA^*$. Technical Report ANL-6456, Argonne National Laboratory, 1961.
[9] G. Hewer, C. Kenney, The sensitivity of the stable Lyapunov equation, SIAM J. Cont. Optim. 26 (1988) 321–344.
[10] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, UK, 1985.
[11] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, UK, 1991.
[12] C. Kenney, G. Hewer, The sensitivity of the algebraic and differential Riccati equations, SIAM J. Cont. Optim. 28 (1990) 50–69.
[13] M. Konstantinov, P. Petkov, D.W. Gu, V. Mehrmann, Sensitivity of general Lyapunov equations, Technical Report 98-15, Department of Engineering, Leicester University, Leicester, UK, August 1998.
[14] M. Konstantinov, M. Stanislavova, P. Petkov, Perturbation bounds and characterisation of the solution of the associated algebraic Riccati equation, Linear Algebra Appl. 285 (1998) 7–31.
[15] M.M. Konstantinov, P.Hr. Petkov, N.D. Christov, The associated algebraic Riccati equation, in: Proc. Third Internat. Conf. on Systems Engr., Wright State University, Dayton, OH, 1984, pp. 530–537.
[16] M.M. Konstantinov, P.Hr. Petkov, N.D. Christov, Perturbation analysis of the discrete Riccati equation, Kybernetica (Prague) 29 (1993) 18–29.
[17] L. Kronecker, Algebraische Reduction der Schaaren bilineare Formen, S.B. Akad. Berlin, 1880, pp. 1225–1237.
[18] V. Mehrmann, The Autonomous Linear Quadratic Control Problem: Theory and Numerical Solution, Lecture Notes in Control and Information Sciences, No. 163, Springer, Heidelberg, 1991.
[19] P.Hr. Petkov, N.D. Christov, M.M. Konstantinov, Computational Methods for Linear Control Systems, Prentice-Hall, Hemel Hempstead, Herts, UK, 1991.

[20] W.E. Roth, The equations $AX - YB = C$ and $AX - XB = C$ in matrices, Proc. Amer. Math. Soc. 3 (1952) 392–396.
[21] D.E. Rutherford, On the solution of the matrix equation $AX + XB = C$, Nederl. Akad. Wetensch. Proc. Ser. A 35 (1932) 53–59.
[22] P. Stein, On the ranges of two functions of positive definite matrices, J. Algebra 2 (1965) 350–353.
[23] P. Stein, A. Pfeffer, On the ranges of two functions of positive definite matrices II, ICC Bull. 6 (1967) 81–86.
[24] G.W. Stewart, Introduction to Matrix Computations, Academic Press, New York, 1973.
[25] J.J. Sylvester, Sur la solution du cas le plus général des équations linéaires en quantités binaires, c'est-à-dire en quaternions ou en matrices du second ordre, C.R. Acad. Sci. Paris 99 (1884) 117, 118.
[26] J.J. Sylvester, Sur l'équation en matrices $px = xq$, C.R. Acad. Sci. Paris 99 (1884) 67–71.
[27] O. Taussky, H. Wielandt, On the matrix function $AX + X'A'$, Arch. Rat. Mech. Anal. 9 (1962) 93–96.
[28] D.S. Watkins, Fundamentals of Matrix Computations, Wiley, New York, 1991.
[29] J.H.M. Wedderburn, Note on the linear matrix equation, Proc. Edinburgh Math. Soc. 22 (1904) 49–53.
[30] W.M. Wonham, Linear Multivariable Control: A Geometric Approach, second ed., Springer, New York, 1979.